Functions and Their Graphs

1

Choosing a Cellular Telephone Plan

Most consumers choose a cellular telephone provider first and then select an appropriate plan from that provider. What type of plan to select depends on your use of the phone. For example, is text messaging important? How many minutes do you plan to use the phone? Do you desire a data plan to browse the Web? The mathematics learned in this chapter can help you decide which plan is best suited for your particular needs.

—See the Internet-based Chapter Project—

<A Look Back

So far, our discussion has focused on techniques for graphing equations containing two variables.

A Look Ahead>

In this chapter, we look at a special type of equation involving two variables called a *function*. This chapter deals with what a function is, how to graph functions, properties of functions, and how functions are used in applications. The word *function* apparently was introduced by René Descartes in 1637. For him, a function was simply any positive integral power of a variable x. Gottfried Wilhelm Leibniz (1646–1716), who always emphasized the geometric side of mathematics, used the word *function* to denote any quantity associated with a curve, such as the coordinates of a point on the curve. Leonhard Euler (1707–1783) employed the word to mean any equation or formula involving variables and constants. His idea of a function is similar to the one most often seen in courses that precede calculus. Later, the use of functions in investigating heat flow equations led to a very broad definition that originated with Lejeune Dirichlet (1805–1859), which describes a function as a correspondence between two sets. That is the definition used in this text.

Outline

1.1 Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Intervals (Appendix A, Section A.10, pp. A81–A82)
- Solving Inequalities (Appendix A, Section A.10, pp. A84–A86)
- Evaluating Algebraic Expressions, Domain of a Variable (Appendix A, Section A.1, pp. A6–A7)

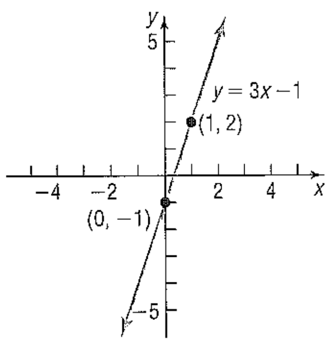
Now Work the 'Are You Prepared?' problems on page 53.

OBJECTIVES

1 Determine Whether a Relation Represents a Function

Often there are situations where the value of one variable is somehow linked to the value of another variable. For example, an individual's level of education is linked to annual income. Engine size is linked to gas mileage. When the value of one variable is related to the value of a second variable, we have a *relation*. A **relation** is a correspondence between two sets. If x and y are two elements in these sets, and if a relation exists between x and y, then we say that x **corresponds** to y or that y **depends on** x, and we write $x \to y$.

Figure 1



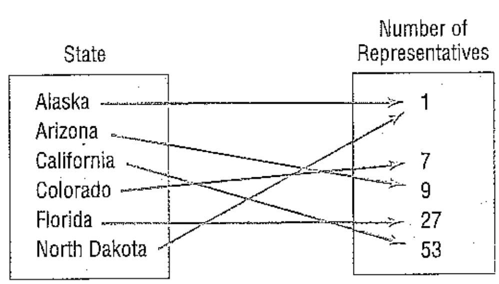
There are a number of ways to express relations between two sets. For example, the equation $y = 3x - 1$ shows a relation between x and y. It says that if we take some number x, multiply it by 3, and then subtract 1, we obtain the corresponding value of y. In this sense, x serves as the **input** to the relation and y is the **output** of the relation. This relation, expressed as a graph is shown in Figure 1.

In addition to being expressed in equations and graphs, relations can be expressed through a technique called *mapping*. A **map** illustrates a relation as a set of inputs with an arrow drawn from each element in the set of inputs to the corresponding element in the set of outputs. **Ordered pairs** can be used to represent $x \to y$ as (x, y).

EXAMPLE 1 Maps and Ordered Pairs as Relations

Figure 2 shows a relation between states and the number of representatives each state has in the House of Representatives (Source: *www.house.gov*). The relation might be named "number of representatives."

Figure 2



In this relation, Alaska corresponds to 1, Arizona corresponds to 9, and so on. Using ordered pairs, this relation would be expressed as

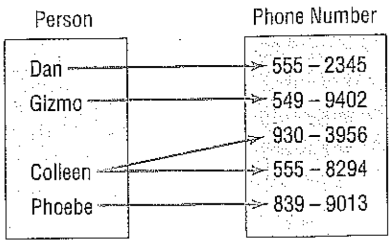
$$\{(\text{Alaska}, 1), (\text{Arizona}, 9), (\text{California}, 53), (\text{Colorado}, 7), (\text{Florida}, 27), (\text{North Dakota}, 1)\}$$

One of the most important concepts in algebra is the *function*. A function is a special type of relation. To understand the idea behind a function, let's revisit the relation presented in Example 1. If we were to ask, "How many representatives does

Alaska have?," you would respond "1." In fact, each input *state* corresponds to a single output *number of representatives*.

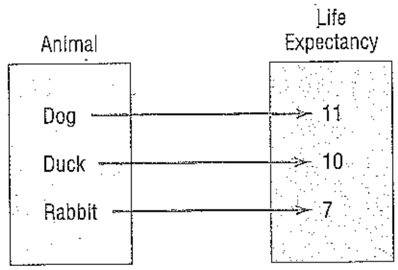
Let's consider a second relation, one that involves a correspondence between four people and their phone numbers. See Figure 3. Notice that Colleen has two telephone numbers. There is no single answer to the question "What is Colleen's phone number?"

Figure 3



Let's look at one more relation. Figure 4 is a relation that shows a correspondence between type of *animal* and *life expectancy*. If asked to determine the life expectancy of a dog, we would all respond "11 years." If asked to determine the life expectancy of a rabbit, we would all respond "7 years."

Figure 4

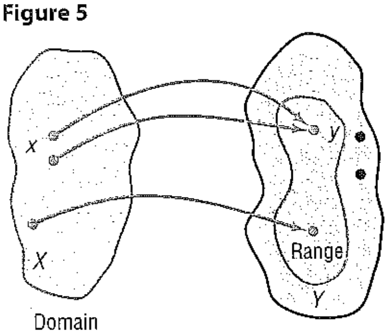


Notice that the relations presented in Figures 2 and 4 have something in common. What is it? In both of these relations, each input corresponds to exactly one output. This leads to the definition of a *function*.

DEFINITION

Let X and Y be two nonempty sets.* A **function** from X into Y is a relation that associates with each element of X exactly one element of Y.

Figure 5



The set X is called the **domain** of the function. For each element x in X, the corresponding element y in Y is called the **value** of the function at x, or the **image** of x. The set of all images of the elements in the domain is called the **range** of the function. See Figure 5.

Since there may be some elements in Y that are not the image of some x in X, it follows that the range of a function may be a subset of Y, as shown in Figure 5. For example, consider the function $y = x^2$. Since $x^2 \geq 0$ for all real numbers x, the range of $y = x^2$ is $\{y | y \geq 0\}$, which is a subset of the set of all real numbers, Y.

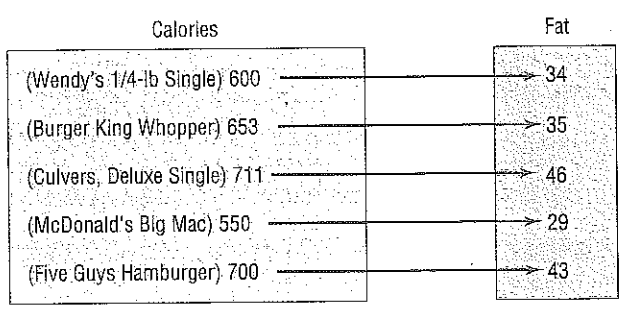
Not all relations between two sets are functions. The next example shows how to determine whether a relation is a function.

EXAMPLE 2 Determining Whether a Relation Represents a Function

Determine which of the following relations represent a function. If the relation is a function, then state its domain and range.

(a) See Figure 6. For this relation, the domain represents the number of calories in a fast-food sandwich, and the range represents the fat content (in grams).

Figure 6
Source: Each company's Web site.



*The sets X and Y will usually be sets of real numbers, in which case a (real) function results. The two sets can also be sets of complex numbers, and then we have defined a complex function. In the broad definition (proposed by Lejeune Dirichlet), X and Y can be any two sets.

(b) See Figure 7. For this relation, the domain represents gasoline stations in Harris County, Texas and the range represents the price per gallon of regular unleaded gasoline in February 2013.

(c) See Figure 8. For this relation, the domain represents the weight (in carats) of pear-cut diamonds, and the range represents the price (in dollars).

Figure 7

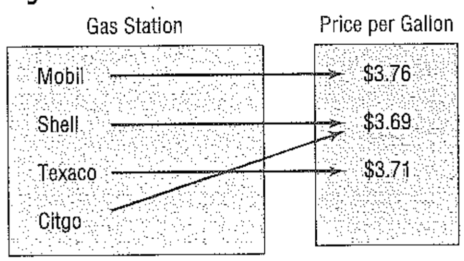
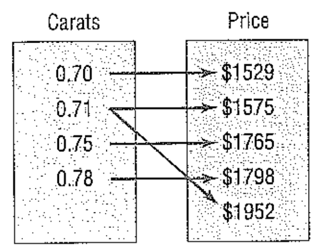


Figure 8

Source: Used with permission of Diamonds.com



Solution (a) The relation in Figure 6 is a function because each element in the domain corresponds to exactly one element in the range. The domain of the function is $\{550, 600, 653, 700, 711\}$, and the range of the function is $\{29, 34, 35, 43, 46\}$.

(b) The relation in Figure 7 is a function because each element in the domain corresponds to exactly one element in the range. The domain of the function is {Mobil, Shell, Texaco, Citgo}. The range of the function is $\{3.69, 3.71, 3.76\}$. Notice that it is okay for more than one element in the domain to correspond to the same element in the range (Shell and Citgo both sell gas for \$3.69 a gallon).

(c) The relation in Figure 8 is not a function because each element in the domain does not correspond to exactly one element in the range. If a 0.71-carat diamond is chosen from the domain, a single price cannot be assigned to it.

Now Work PROBLEM 15

The idea behind a function is its predictability. If the input is known, we can use the function to determine the output. With "nonfunctions," we don't have this predictability. Look back at Figure 6. If asked, "How many grams of fat are in a 600-calorie sandwich?" we could use the correspondence to answer "34." Now consider Figure 8. If asked, "What is the price of a 0.71-carat diamond?" we could not give a single response because two outputs result from the single input "0.71." For this reason, the relation in Figure 8 is not a function.

> **In Words**
> For a function, no input has more than one output. The domain of a function is the set of all inputs; the range is the set of all outputs.

We may also think of a function as a set of ordered pairs (x, y) in which no ordered pairs have the same first element and different second elements. The set of all first elements x is the domain of the function, and the set of all second elements y is its range. Each element x in the domain corresponds to exactly one element y in the range.

EXAMPLE 3 **Determining Whether a Relation Represents a Function**

Determine whether each relation represents a function. If it is a function, state the domain and range.

(a) $\{(1, 4), (2, 5), (3, 6), (4, 7)\}$

(b) $\{(1, 4), (2, 4), (3, 5), (6, 10)\}$

(c) $\{(-3, 9), (-2, 4), (0, 0), (1, 1), (-3, 8)\}$

Solution (a) This relation is a function because there are no ordered pairs with the same first element and different second elements. The domain of this function is $\{1, 2, 3, 4\}$, and its range is $\{4, 5, 6, 7\}$.

(b) This relation is a function because there are no ordered pairs with the same first element and different second elements. The domain of this function is $\{1, 2, 3, 6\}$, and its range is $\{4, 5, 10\}$.

(c) This relation is not a function because there are two ordered pairs, $(-3, 9)$ and $(-3, 8)$, that have the same first element and different second elements.

In Example 3(b), notice that 1 and 2 in the domain both have the same image in the range. This does not violate the definition of a function; two different first elements can have the same second element. A violation of the definition occurs when two ordered pairs have the same first element and different second elements, as in Example 3(c).

Now Work PROBLEM 19

Up to now we have shown how to identify when a relation is a function for relations defined by mappings (Example 2) and ordered pairs (Example 3). But relations can also be expressed as equations. The circumstances under which equations are functions are discussed next.

To determine whether an equation, where y depends on x, is a function, it is often easiest to solve the equation for y. If any value of x in the domain corresponds to more than one y, the equation does not define a function; otherwise, it does define a function.

EXAMPLE 4 **Determining Whether an Equation Is a Function**

Determine whether the equation $y = 2x - 5$ defines y as a function of x.

Solution The equation tells us to take an input x, multiply it by 2, and then subtract 5. For any input x, these operations yield only one output y. For example, if $x = 1$, then $y = 2(1) - 5 = -3$. If $x = 3$, then $y = 2(3) - 5 = 1$. For this reason, the equation is a function.

EXAMPLE 5 **Determining Whether an Equation Is a Function**

Determine whether the equation $x^2 + y^2 = 1$ defines y as a function of x.

Solution To determine whether the equation $x^2 + y^2 = 1$, which defines the unit circle, is a function, solve the equation for y.

$$\begin{aligned} x^2 + y^2 &= 1 \\ y^2 &= 1 - x^2 \\ y &= \pm\sqrt{1 - x^2} \end{aligned}$$

For values of x between -1 and 1, two values of y result. For example, if $x = 0$, then $y = \pm 1$, so two different outputs result from the same input. This means that the equation $x^2 + y^2 = 1$ does not define a function.

Now Work PROBLEM 33

2 Find the Value of a Function

Functions are often denoted by letters such as, f, F, g, G, and others. If f is a function, then for each number x in its domain, the corresponding image in the range is designated by the symbol $f(x)$, read as "f of x" or as "f at x." We refer to $f(x)$ as the **value of f at the number x**; $f(x)$ is the number that results when x is given and the function f is applied; $f(x)$ is the output corresponding to x or the image of x; $f(x)$ does *not* mean "f times x." For example, the function given in Example 4 may be written as $y = f(x) = 2x - 5$. Then $f(1) = -3$ and $f(3) = 1$.

Figure 9 illustrates some other functions. Notice that, in every function, for each x in the domain there is one value in the range.

Figure 9

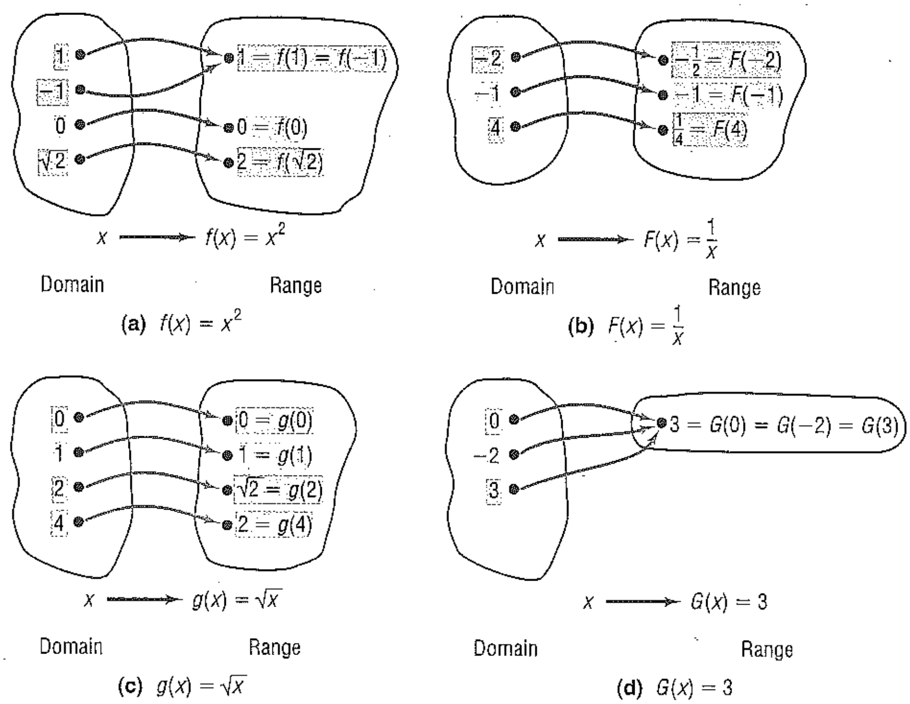
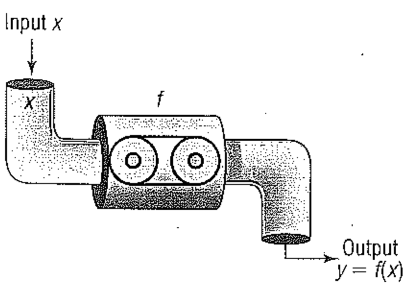


Figure 10



Sometimes it is helpful to think of a function f as a machine that receives as input a number from the domain, manipulates it, and outputs a value. See Figure 10.

The restrictions on this input/output machine are as follows:

1. It accepts only numbers from the domain of the function.
2. For each input, there is exactly one output (which may be repeated for different inputs).

For a function $y = f(x)$, the variable x is called the **independent variable,** because it can be assigned any of the permissible numbers from the domain. The variable y is called the **dependent variable,** because its value depends on x.

Any symbols can be used to represent the independent and dependent variables. For example, if f is the *cube function,* then f can be given by $f(x) = x^3$ or $f(t) = t^3$ or $f(z) = z^3$. All three functions are the same. Each says to cube the independent variable to get the output. In practice, the symbols used for the independent and dependent variables are based on common usage, such as using C for cost in business.

The independent variable is also called the **argument** of the function. Thinking of the independent variable as an argument can sometimes make it easier to find the value of a function. For example, if f is the function defined by $f(x) = x^3$, then f tells us to cube the argument. Thus, $f(2)$ means to cube 2, $f(a)$ means to cube the number a, and $f(x + h)$ means to cube the quantity $x + h$.

EXAMPLE 6

Finding Values of a Function

For the function f defined by $f(x) = 2x^2 - 3x$, evaluate

(a) $f(3)$ (b) $f(x) + f(3)$ (c) $3f(x)$ (d) $f(-x)$

(e) $-f(x)$ (f) $f(3x)$ (g) $f(x + 3)$ (h) $\dfrac{f(x + h) - f(x)}{h}$ $h \neq 0$

Solution (a) Substitute 3 for x in the equation for f, $f(x) = 2x^2 - 3x$, to get

$$f(3) = 2(3)^2 - 3(3) = 18 - 9 = 9$$

The image of 3 is 9.

(b) $f(x) + f(3) = (2x^2 - 3x) + (9) = 2x^2 - 3x + 9$

(c) Multiply the equation for f by 3.

$$3f(x) = 3(2x^2 - 3x) = 6x^2 - 9x$$

(d) Substitute $-x$ for x in the equation for f and simplify.

$$f(-x) = 2(-x)^2 - 3(-x) = 2x^2 + 3x \quad \text{Notice the use of parentheses here.}$$

(e) $-f(x) = -(2x^2 - 3x) = -2x^2 + 3x$

(f) Substitute $3x$ for x in the equation for f and simplify.

$$f(3x) = 2(3x)^2 - 3(3x) = 2(9x^2) - 9x = 18x^2 - 9x$$

(g) Substitute $x + 3$ for x in the equation for f and simplify.

$$\begin{aligned} f(x+3) &= 2(x+3)^2 - 3(x+3) \\ &= 2(x^2 + 6x + 9) - 3x - 9 \\ &= 2x^2 + 12x + 18 - 3x - 9 \\ &= 2x^2 + 9x + 9 \end{aligned}$$

(h) $$\frac{f(x+h) - f(x)}{h} = \frac{[2(x+h)^2 - 3(x+h)] - [2x^2 - 3x]}{h}$$

$\uparrow$ $f(x+h) = 2(x+h)^2 - 3(x+h)$

$$= \frac{2(x^2 + 2xh + h^2) - 3x - 3h - 2x^2 + 3x}{h} \quad \text{Simplify.}$$

$$= \frac{2x^2 + 4xh + 2h^2 - 3h - 2x^2}{h} \quad \text{Distribute and combine like terms.}$$

$$= \frac{4xh + 2h^2 - 3h}{h} \quad \text{Combine like terms.}$$

$$= \frac{h(4x + 2h - 3)}{h} \quad \text{Factor out } h.$$

$$= 4x + 2h - 3 \quad \text{Divide out the } h\text{'s.}$$

Notice in this example that $f(x + 3) \neq f(x) + f(3)$, $f(-x) \neq -f(x)$, and $3f(x) \neq f(3x)$.

The expression in part (h) is called the **difference quotient** of f, an important expression in calculus.

Now Work PROBLEMS 39 AND 75

Most calculators have special keys that allow you to find the value of certain commonly used functions. For example, you should be able to find the square function $f(x) = x^2$, the square root function $f(x) = \sqrt{x}$, the reciprocal function $f(x) = \frac{1}{x} = x^{-1}$, and many others that will be discussed later in this text (such as $\ln x$ and $\log x$). Verify the results of Example 7, which follows, on your calculator.

EXAMPLE 7 Finding Values of a Function on a Calculator

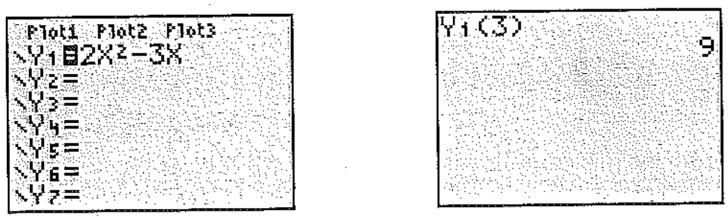
(a) $f(x) = x^2$ $\quad f(1.234) = 1.234^2 = 1.522756$

(b) $F(x) = \frac{1}{x}$ $\quad F(1.234) = \frac{1}{1.234} \approx 0.8103727715$

(c) $g(x) = \sqrt{x}$ $\quad g(1.234) = \sqrt{1.234} \approx 1.110855526$

COMMENT Graphing calculators can be used to evaluate any function. Figure 11 shows the result obtained in Example 6(a) on a TI-84 Plus graphing calculator with the function to be evaluated, $f(x) = 2x^2 - 3x$, in Y_1.

Figure 11



COMMENT The explicit form of a function is the form required by a graphing calculator.

Implicit Form of a Function

In general, when a function f is defined by an equation in x and y, we say that the function f is given **implicitly.** If it is possible to solve the equation for y in terms of x, then we write $y = f(x)$ and say that the function is given **explicitly.** For example,

Implicit Form	**Explicit Form**
$3x + y = 5$	$y = f(x) = -3x + 5$
$x^2 - y = 6$	$y = f(x) = x^2 - 6$
$xy = 4$	$y = f(x) = \frac{4}{x}$

SUMMARY

Important Facts about Functions

(a) For each x in the domain of a function f, there is exactly one image $f(x)$ in the range; however, an element in the range can result from more than one x in the domain.

(b) f is the symbol that we use to denote the function. It is symbolic of the equation (rule) that we use to get from an x in the domain to $f(x)$ in the range.

(c) If $y = f(x)$, then x is called the independent variable or argument of f, and y is called the dependent variable or the value of f at x.

3 Find the Domain of a Function Defined by an Equation

Often the domain of a function f is not specified; instead, only the equation defining the function is given. In such cases, we agree that the **domain of** f is the largest set of real numbers for which the value $f(x)$ is a real number. The domain of a function f is the same as the domain of the variable x in the expression $f(x)$.

EXAMPLE 8 Finding the Domain of a Function

Find the domain of each of the following functions.

(a) $f(x) = x^2 + 5x$ (b) $g(x) = \dfrac{3x}{x^2 - 4}$

(c) $h(t) = \sqrt{4 - 3t}$ (d) $F(x) = \dfrac{\sqrt{3x + 12}}{x - 5}$

Solution

(a) The function says to square a number and then add five times the number. Since these operations can be performed on any real number, the domain of f is the set of all real numbers.

(b) The function g says to divide $3x$ by $x^2 - 4$. Since division by 0 is not defined, the denominator $x^2 - 4$ can never be 0, so x can never equal -2 or 2. The domain of the function g is $\{x \mid x \neq -2, x \neq 2\}$.

In Words

The domain of g found in Example 8(b) is $\{x|x \neq -2, x \neq 2\}$. This notation is read, "The domain of the function g is the set of all real numbers x such that x does not equal -2 and x does not equal 2."

(c) The function h says to take the square root of $4 - 3t$. But only nonnegative numbers have real square roots, so the expression under the square root (the radicand) must be nonnegative (greater than or equal to zero). This requires that

$$4 - 3t \geq 0$$
$$-3t \geq -4$$
$$t \leq \frac{4}{3}$$

The domain of h is $\left\{t \middle| t \leq \frac{4}{3}\right\}$ or the interval $\left(-\infty, \frac{4}{3}\right]$.

(d) The function F says to take the square root of $3x + 12$ and divide this result by $x - 5$. This requires that $3x + 12 \geq 0$, so $x \geq -4$, and that $x - 5 \neq 0$, so $x \neq 5$. Combining these two restrictions, the domain of F is $\{x|x \geq -4, x \neq 5\}$.

The following steps may prove helpful for finding the domain of a function that is defined by an equation and whose domain is a subset of the real numbers.

Finding the Domain of a Function Defined by an Equation

1. Start with the domain as the set of real numbers.
2. If the equation has a denominator, exclude any numbers that give a zero denominator.
3. If the equation has a radical of even index, exclude any numbers that cause the expression inside the radical (the *radicand*) to be negative. That is, solve *radicand* ≥ 0.

Now Work PROBLEM 51

If x is in the domain of a function f, we shall say that **f is defined at x,** or **$f(x)$ exists.** If x is not in the domain of f, we say that **f is not defined at x,** or **$f(x)$ does not exist.** For example, if $f(x) = \frac{x}{x^2 - 1}$, then $f(0)$ exists, but $f(1)$ and $f(-1)$ do not exist. (Do you see why?)

We will say more about finding the range when we look at the graph of a function in the next section. When a function is defined by an equation, it can be difficult to find the range. Therefore, we shall usually be content to find just the domain of a function when the function is defined by an equation. We shall express the domain of a function using inequalities, interval notation, set notation, or words, whichever is most convenient.

When functions are used in applications, the domain may be restricted by physical or geometric considerations. For example, the domain of the function f defined by $f(x) = x^2$ is the set of all real numbers. However, if f is used to obtain the area of a square when the length x of a side is known, then we must restrict the domain of f to the positive real numbers, since the length of a side can never be 0 or negative.

EXAMPLE 9 **Finding the Domain in an Application**

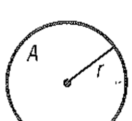
Express the area of a circle as a function of its radius. Find the domain.

Solution See Figure 12. The formula for the area A of a circle of radius r is $A = \pi r^2$. Using r to represent the independent variable and A to represent the dependent variable, the function expressing this relationship is

$$A(r) = \pi r^2$$

In this setting, the domain is $\{r|r > 0\}$. (Do you see why?)

Figure 12



Observe, in the solution to Example 9, that the symbol A is used in two ways: It is used to name the function, and it is used to symbolize the dependent variable. This double use is common in applications and should not cause any difficulty.

Now Work PROBLEM 89

4 Form the Sum, Difference, Product, and Quotient of Two Functions

Next we introduce some operations on functions. Functions, like numbers, can be added, subtracted, multiplied, and divided. For example, if $f(x) = x^2 + 9$ and $g(x) = 3x + 5$, then

$$f(x) + g(x) = (x^2 + 9) + (3x + 5) = x^2 + 3x + 14$$

The new function $y = x^2 + 3x + 14$ is called the *sum function* $f + g$. Similarly,

$$f(x) \cdot g(x) = (x^2 + 9)(3x + 5) = 3x^3 + 5x^2 + 27x + 45$$

The new function $y = 3x^3 + 5x^2 + 27x + 45$ is called the *product function* $f \cdot g$.

The general definitions are given next.

DEFINITION

If f and g are functions:
The **sum** $f + g$ is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

In Words
Remember, the symbol $\cap$ stands for intersection. It means you should find the elements that are common to two sets.

The domain of $f + g$ consists of the numbers x that are in the domains of both f and g. That is, domain of $f + g$ = domain of $f \cap$ domain of g.

DEFINITION

The **difference** $f - g$ is the function defined by

$$(f - g)(x) = f(x) - g(x)$$

The domain of $f - g$ consists of the numbers x that are in the domains of both f and g. That is, domain of $f - g$ = domain of $f \cap$ domain of g.

DEFINITION

The **product** $f \cdot g$ is the function defined by

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

The domain of $f \cdot g$ consists of the numbers x that are in the domains of both f and g. That is, domain of $f \cdot g$ = domain of $f \cap$ domain of g.

DEFINITION

The **quotient** $\dfrac{f}{g}$ is the function defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad g(x) \neq 0$$

The domain of $\dfrac{f}{g}$ consists of the numbers x for which $g(x) \neq 0$ and that are in the domains of both f and g. That is,

$$\text{domain of } \frac{f}{g} = \{x \mid g(x) \neq 0\} \cap \text{ domain of } f \cap \text{ domain of } g$$

EXAMPLE 10 **Operations on Functions**

Let f and g be two functions defined as

$$f(x) = \frac{1}{x+2} \quad \text{and} \quad g(x) = \frac{x}{x-1}$$

Find the following, and determine the domain in each case.

(a) $(f+g)(x)$ (b) $(f-g)(x)$ (c) $(f \cdot g)(x)$ (d) $\left(\dfrac{f}{g}\right)(x)$

Solution The domain of f is $\{x \mid x \neq -2\}$ and the domain of g is $\{x \mid x \neq 1\}$.

(a)
$$(f+g)(x) = f(x) + g(x) = \frac{1}{x+2} + \frac{x}{x-1}$$
$$= \frac{x-1}{(x+2)(x-1)} + \frac{x(x+2)}{(x+2)(x-1)} = \frac{x^2+3x-1}{(x+2)(x-1)}$$

The domain of $f + g$ consists of those numbers x that are in the domains of both f and g. Therefore, the domain of $f + g$ is $\{x \mid x \neq -2, x \neq 1\}$.

(b)
$$(f-g)(x) = f(x) - g(x) = \frac{1}{x+2} - \frac{x}{x-1}$$
$$= \frac{x-1}{(x+2)(x-1)} - \frac{x(x+2)}{(x+2)(x-1)} = \frac{-(x^2+x+1)}{(x+2)(x-1)}$$

The domain of $f - g$ consists of those numbers x that are in the domains of both f and g. Therefore, the domain of $f - g$ is $\{x \mid x \neq -2, x \neq 1\}$.

(c)
$$(f \cdot g)(x) = f(x) \cdot g(x) = \frac{1}{x+2} \cdot \frac{x}{x-1} = \frac{x}{(x+2)(x-1)}$$

The domain of $f \cdot g$ consists of those numbers x that are in the domains of both f and g. Therefore, the domain of $f \cdot g$ is $\{x \mid x \neq -2, x \neq 1\}$.

(d)
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\dfrac{1}{x+2}}{\dfrac{x}{x-1}} = \frac{1}{x+2} \cdot \frac{x-1}{x} = \frac{x-1}{x(x+2)}$$

The domain of $\dfrac{f}{g}$ consists of the numbers x for which $g(x) \neq 0$ and that are in the domains of both f and g. Since $g(x) = 0$ when $x = 0$, we exclude 0 as well as -2 and 1 from the domain. The domain of $\dfrac{f}{g}$ is $\{x \mid x \neq -2, x \neq 0, x \neq 1\}$.

Now Work PROBLEM 63

In calculus, it is sometimes helpful to view a complicated function as the sum, difference, product, or quotient of simpler functions. For example,

$F(x) = x^2 + \sqrt{x}$ is the sum of $f(x) = x^2$ and $g(x) = \sqrt{x}$.

$H(x) = \dfrac{x^2 - 1}{x^2 + 1}$ is the quotient of $f(x) = x^2 - 1$ and $g(x) = x^2 + 1$.

SUMMARY

Function	A relation between two sets of real numbers so that each number x in the first set, the domain, has corresponding to it exactly one number y in the second set. A set of ordered pairs (x, y) or $(x, f(x))$ in which no first element is paired with two different second elements. The range is the set of y-values of the function that are the images of the x-values in the domain. A function f may be defined implicitly by an equation involving x and y or explicitly by writing $y = f(x)$.
Unspecified domain	If a function f is defined by an equation and no domain is specified, then the domain will be taken to be the largest set of real numbers for which the equation defines a real number.
Function notation	$y = f(x)$ f is a symbol for the function. x is the independent variable or argument. y is the dependent variable. $f(x)$ is the value of the function at x, or the image of x.

1.1 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The inequality $-1 < x < 3$ can be written in interval notation as ________. (pp. A81–A82)
2. If $x = -2$, the value of the expression $3x^2 - 5x + \frac{1}{x}$ is ______. (pp. A6–A7)
3. The domain of the variable in the expression $\frac{x-3}{x+4}$ is ________. (p. A7)
4. Solve the inequality $3 - 2x > 5$. Graph the solution set. (pp. A84–A86)

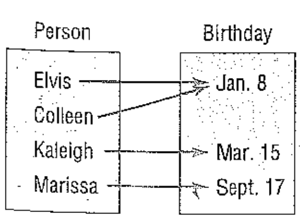
Concepts and Vocabulary

5. If f is a function defined by the equation $y = f(x)$, then x is called the __________ variable and y is the __________ variable.
6. The set of all images of the elements in the domain of a function is called the _______.
7. If the domain of f is all real numbers in the interval $[0, 7]$ and the domain of g is all real numbers in the interval $[-2, 5]$, then the domain of $f + g$ is all real numbers in the interval _____.
8. The domain of $\frac{f}{g}$ consists of numbers x for which $g(x)$ ____ 0 that are in the domains of both ____ and ____.
9. If $f(x) = x + 1$ and $g(x) = x^3$, then __________ $= x^3 - (x + 1)$.
10. ***True or False*** Every relation is a function.
11. ***True or False*** The domain of $(f \cdot g)(x)$ consists of the numbers x that are in the domains of both f and g.
12. ***True or False*** The independent variable is sometimes referred to as the argument of the function.
13. ***True or False*** If no domain is specified for a function f, then the domain of f is taken to be the set of real numbers.
14. ***True or False*** The domain of the function $f(x) = \frac{x^2 - 4}{x}$ is $\{x \mid x \neq \pm 2\}$.

Skill Building

In Problems 15–26, determine whether each relation represents a function. For each function, state the domain and range.

15.



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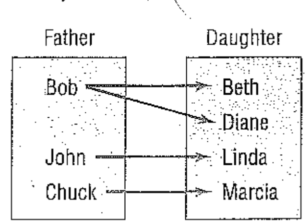
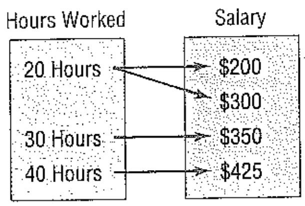
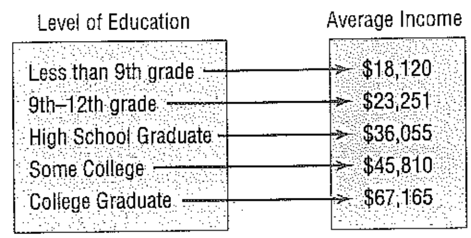


17. 

18. 

19. $\{(2, 6), (-3, 6), (4, 9), (2, 10)\}$
20. $\{(-2, 5), (-1, 3), (3, 7), (4, 12)\}$
21. $\{(1, 3), (2, 3), (3, 3), (4, 3)\}$
22. $\{(0, -2), (1, 3), (2, 3), (3, 7)\}$
23. $\{(-2, 4), (-2, 6), (0, 3), (3, 7)\}$
24. $\{(-4, 4), (-3, 3), (-2, 2), (-1, 1), (-4, 0)\}$
25. $\{(-2, 4), (-1, 1), (0, 0), (1, 1)\}$
26. $\{(-2, 16), (-1, 4), (0, 3), (1, 4)\}$

In Problems 27–38, determine whether the equation defines y as a function of x.

27. $y = x^2$
28. $y = x^3$
29. $y = \frac{1}{x}$
30. $y = |x|$
31. $y^2 = 4 - x^2$
32. $y = \pm\sqrt{1 - 2x}$
33. $x = y^2$
34. $x + y^2 = 1$
35. $y = 2x^2 - 3x + 4$
36. $y = \frac{3x - 1}{x + 2}$
37. $2x^2 + 3y^2 = 1$
38. $x^2 - 4y^2 = 1$

In Problems 39–46, find the following for each function:

(a) $f(0)$ (b) $f(1)$ (c) $f(-1)$ (d) $f(-x)$ (e) $-f(x)$ (f) $f(x + 1)$ (g) $f(2x)$ (h) $f(x + h)$

39. $f(x) = 3x^2 + 2x - 4$
40. $f(x) = -2x^2 + x - 1$
41. $f(x) = \frac{x}{x^2 + 1}$
42. $f(x) = \frac{x^2 - 1}{x + 4}$
43. $f(x) = |x| + 4$
44. $f(x) = \sqrt{x^2 + x}$
45. $f(x) = \frac{2x + 1}{3x - 5}$
46. $f(x) = 1 - \frac{1}{(x + 2)^2}$

In Problems 47–62, find the domain of each function.

47. $f(x) = -5x + 4$
48. $f(x) = x^2 + 2$
49. $f(x) = \frac{x}{x^2 + 1}$
50. $f(x) = \frac{x^2}{x^2 + 1}$
51. $g(x) = \frac{x}{x^2 - 16}$
52. $h(x) = \frac{2x}{x^2 - 4}$
53. $F(x) = \frac{x - 2}{x^3 + x}$
54. $G(x) = \frac{x + 4}{x^3 - 4x}$
55. $h(x) = \sqrt{3x - 12}$
56. $G(x) = \sqrt{1 - x}$
57. $f(x) = \frac{4}{\sqrt{x - 9}}$
58. $f(x) = \frac{x}{\sqrt{x - 4}}$
59. $p(x) = \sqrt{\frac{2}{x - 1}}$
60. $q(x) = \sqrt{-x - 2}$
61. $P(t) = \frac{\sqrt{t - 4}}{3t - 21}$
62. $h(z) = \frac{\sqrt{z + 3}}{z - 2}$

In Problems 63–72, for the given functions f and g, find the following. For parts (a)–(d), also find the domain.

(a) $(f + g)(x)$ (b) $(f - g)(x)$ (c) $(f \cdot g)(x)$ (d) $\left(\frac{f}{g}\right)(x)$

(e) $(f + g)(3)$ (f) $(f - g)(4)$ (g) $(f \cdot g)(2)$ (h) $\left(\frac{f}{g}\right)(1)$

63. $f(x) = 3x + 4;\quad g(x) = 2x - 3$
64. $f(x) = 2x + 1;\quad g(x) = 3x - 2$
65. $f(x) = x - 1;\quad g(x) = 2x^2$
66. $f(x) = 2x^2 + 3;\quad g(x) = 4x^3 + 1$
67. $f(x) = \sqrt{x};\quad g(x) = 3x - 5$
68. $f(x) = |x|;\quad g(x) = x$
69. $f(x) = 1 + \frac{1}{x};\quad g(x) = \frac{1}{x}$
70. $f(x) = \sqrt{x - 1};\quad g(x) = \sqrt{4 - x}$
71. $f(x) = \frac{2x + 3}{3x - 2};\quad g(x) = \frac{4x}{3x - 2}$
72. $f(x) = \sqrt{x + 1};\quad g(x) = \frac{2}{x}$

73. Given $f(x) = 3x + 1$ and $(f + g)(x) = 6 - \frac{1}{2}x$, find the function g.

74. Given $f(x) = \frac{1}{x}$ and $\left(\frac{f}{g}\right)(x) = \frac{x + 1}{x^2 - x}$, find the function g.

In Problems 75–82, find the difference quotient of f; that is, find $\dfrac{f(x+h)-f(x)}{h}$, $h \neq 0$, *for each function. Be sure to simplify.*

75. $f(x) = 4x + 3$
76. $f(x) = -3x + 1$
77. $f(x) = x^2 - x + 4$
78. $f(x) = 3x^2 - 2x + 6$
79. $f(x) = \dfrac{1}{x^2}$
80. $f(x) = \dfrac{1}{x+3}$
81. $f(x) = \sqrt{x}$ [**Hint:** Rationalize the numerator.]
82. $f(x) = \sqrt{x+1}$

Applications and Extensions

83. If $f(x) = 2x^3 + Ax^2 + 4x - 5$ and $f(2) = 5$, what is the value of A?

84. If $f(x) = 3x^2 - Bx + 4$ and $f(-1) = 12$, what is the value of B?

85. If $f(x) = \dfrac{3x+8}{2x-A}$ and $f(0) = 2$, what is the value of A?

86. If $f(x) = \dfrac{2x-B}{3x+4}$ and $f(2) = \dfrac{1}{2}$, what is the value of B?

87. If $f(x) = \dfrac{2x-A}{x-3}$ and $f(4) = 0$, what is the value of A? Where is f not defined?

88. If $f(x) = \dfrac{x-B}{x-A}$, $f(2) = 0$, and $f(1)$ is undefined, what are the values of A and B?

89. **Geometry** Express the area A of a rectangle as a function of the length x if the length of the rectangle is twice its width.

90. **Geometry** Express the area A of an isosceles right triangle as a function of the length x of one of the two equal sides.

91. **Constructing Functions** Express the gross salary G of a person who earns \$10 per hour as a function of the number x of hours worked.

92. **Constructing Functions** Tiffany, a commissioned sales person, earns \$100 base pay plus \$10 per item sold. Express her gross salary G as a function of the number x of items sold.

93. **Population as a Function of Age** The function

$$P(a) = 0.004a^2 - 3.792a + 317.946$$

represents the population P (in millions) of Americans that were a years of age or older in 2011.

Source: U.S. Census Bureau

(a) Identify the dependent and independent variables.
(b) Evaluate $P(20)$. Provide a verbal explanation of the meaning of $P(20)$.
(c) Evaluate $P(0)$. Provide a verbal explanation of the meaning of $P(0)$.

94. **Number of Rooms** The function

$$N(r) = -2.08r^2 + 22.901r - 36.06$$

represents the number N of housing units (in millions) in 2011 that had r rooms, where r is an integer and $2 \le r \le 9$.

Source: U.S. Census Bureau

(a) Identify the dependent and independent variables.
(b) Evaluate $N(3)$. Provide a verbal explanation of the meaning of $N(3)$.

95. **Effect of Gravity on Earth** If a rock falls from a height of 20 meters on Earth, the height H (in meters) after x seconds is approximately

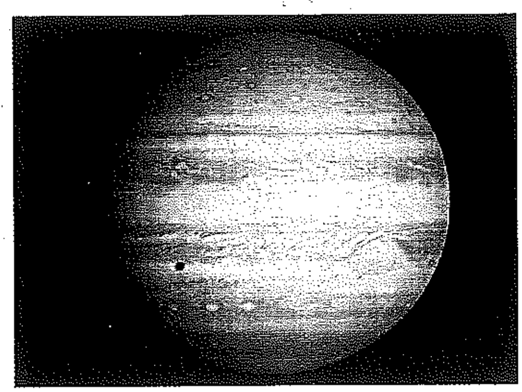
$$H(x) = 20 - 4.9x^2$$

(a) What is the height of the rock when $x = 1$ second? $x = 1.1$ seconds? $x = 1.2$ seconds? $x = 1.3$ seconds?
(b) When is the height of the rock 15 meters? When is it 10 meters? When is it 5 meters?
(c) When does the rock strike the ground?

96. **Effect of Gravity on Jupiter** If a rock falls from a height of 20 meters on the planet Jupiter, its height H (in meters) after x seconds is approximately

$$H(x) = 20 - 13x^2$$

(a) What is the height of the rock when $x = 1$ second? $x = 1.1$ seconds? $x = 1.2$ seconds?
(b) When is the height of the rock 15 meters? When is it 10 meters? When is it 5 meters?
(c) When does the rock strike the ground?



97. **Cost of Trans-Atlantic Travel** A Boeing 747 crosses the Atlantic Ocean (3000 miles) with an airspeed of 500 miles per hour. The cost C (in dollars) per passenger is given by

$$C(x) = 100 + \frac{x}{10} + \frac{36{,}000}{x}$$

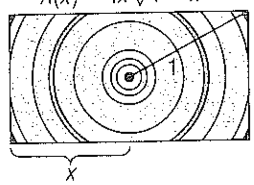
where x is the ground speed (airspeed $\pm$ wind).

(a) What is the cost per passenger for quiescent (no wind) conditions?
(b) What is the cost per passenger with a head wind of 50 miles per hour?
(c) What is the cost per passenger with a tail wind of 100 miles per hour?
(d) What is the cost per passenger with a head wind of 100 miles per hour?

98. **Cross-sectional Area** The cross-sectional area of a beam cut from a log with radius 1 foot is given by the function $A(x) = 4x\sqrt{1-x^2}$, where x represents the length, in feet, of half the base of the beam. See the figure on the next page. Determine the cross-sectional area of the beam if the length of half the base of the beam is as follows:

(a) One-third of a foot
(b) One-half of a foot
(c) Two-thirds of a foot

$A(x) = 4x\sqrt{1 - x^2}$



99. Economics The **participation rate** is the number of people in the labor force divided by the civilian population (excludes military). Let $L(x)$ represent the size of the labor force in year x and $P(x)$ represent the civilian population in year x. Determine a function that represents the participation rate R as a function of x.

100. Crimes Suppose that $V(x)$ represents the number of violent crimes committed in year x and $P(x)$ represents the number of property crimes committed in year x. Determine a function T that represents the combined total of violent crimes and property crimes in year x.

101. Health Care Suppose that $P(x)$ represents the percentage of income spent on health care in year x and $I(x)$ represents income in year x. Determine a function H that represents total health care expenditures in year x.

102. Income Tax Suppose that $I(x)$ represents the income of an individual in year x before taxes and $T(x)$ represents the individual's tax bill in year x. Determine a function N that represents the individual's net income (income after taxes) in year x.

103. Profit Function Suppose that the revenue R, in dollars, from selling x cell phones, in hundreds, is $R(x) = -1.2x^2 + 220x$. The cost C, in dollars, of selling x cell phones, in hundreds, is $C(x) = 0.05x^3 - 2x^2 + 65x + 500$.
(a) Find the profit function, $P(x) = R(x) - C(x)$.
(b) Find the profit if $x = 15$ hundred cell phones are sold.
(c) Interpret $P(15)$.

104. Profit Function Suppose that the revenue R, in dollars, from selling x clocks is $R(x) = 30x$. The cost C, in dollars, of selling x clocks is $C(x) = 0.1x^2 + 7x + 400$.
(a) Find the profit function, $P(x) = R(x) - C(x)$.
(b) Find the profit if $x = 30$ clocks are sold.
(c) Interpret $P(30)$.

105. Stopping Distance When the driver of a vehicle observes an impediment, the total stopping distance involves both the reaction distance (the distance the vehicle travels while the driver moves his or her foot to the brake pedal) and the braking distance (the distance the vehicle travels once the brakes are applied). For a car traveling at a speed of v miles per hour, the reaction distance R, in feet, can be estimated by $R(v) = 2.2v$. Suppose that the braking distance B, in feet, for a car is given by $B(v) = 0.05v^2 + 0.4v - 15$.
(a) Find the stopping distance function $D(v) = R(v) + B(v)$.
(b) Find the stopping distance if the car is traveling at a speed of 60 mph.
(c) Interpret $D(60)$.

106. Some functions f have the property that $f(a + b) = f(a) + f(b)$ for all real numbers a and b. Which of the following functions have this property?
(a) $h(x) = 2x$ (b) $g(x) = x^2$
(c) $F(x) = 5x - 2$ (d) $G(x) = \frac{1}{x}$

Discussion and Writing

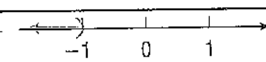
107. Are the functions $f(x) = x - 1$ and $g(x) = \frac{x^2 - 1}{x + 1}$ the same? Explain.

108. Investigate when, historically, the use of the function notation $y = f(x)$ first appeared.

109. Find a function H that multiplies a number x by 3 and then subtracts the cube of x and divides the result by your age.

'Are You Prepared?' Answers

1. $(-1, 3)$ **2.** 21.5 **3.** $\{x \mid x \neq -4\}$ **4.** $\{x \mid x < -1\}$



1.2 The Graph of a Function

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Graphs of Equations (Foundations Section 2, pp. 9–11)
- Intercepts (Foundations Section 2, pp. 11–12)

Now Work the 'Are You Prepared?' problems on page 61.

OBJECTIVES 1 Identify the Graph of a Function (p. 57)
2 Obtain Information from or about the Graph of a Function (p. 58)

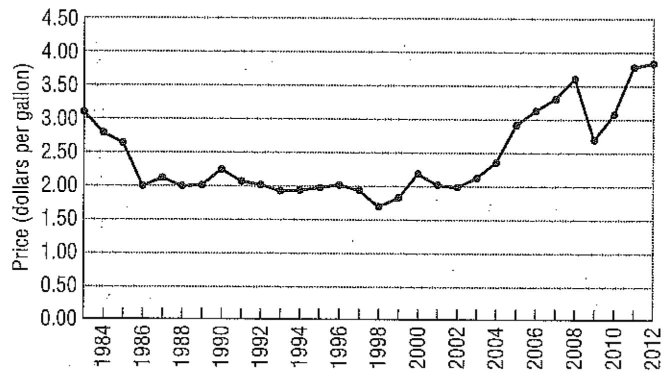
In applications, a graph often demonstrates more clearly the relationship between two variables than, say, an equation or table would. For example, Table 1 on the next page shows the average price of gasoline at a particular gas station in Texas (for the years 1983–2012 adjusted for inflation, based on 2012 dollars). If we plot these data and then connect the points, we obtain Figure 13 (also on the next page).

Table 1

Year	Price	Year	Price	Year	Price
1983	3.10	1993	1.93	2003	2.13
1984	2.79	1994	1.94	2004	2.36
1985	2.64	1995	1.98	2005	2.92
1986	2.00	1996	2.02	2006	3.13
1987	2.12	1997	1.94	2007	3.31
1988	2.00	1998	1.70	2008	3.61
1989	2.01	1999	1.83	2009	2.70
1990	2.25	2000	2.19	2010	3.08
1991	2.07	2001	2.02	2011	3.78
1992	2.02	2002	1.99	2012	3.84

Source: http://www.randomuseless.info/gasprice/gasprice.html

Figure 13 Average retail price of gasoline (2012 dollars)



Source: http://www.randomuseless.info/gasprice/gasprice.html

We can see from the graph that the price of gasoline (adjusted for inflation) fell from 1983 to 1986, stayed roughly the same from 1986 to 2002, and rose rapidly from 2002 to 2008. The graph also shows that the lowest price occurred in 1998. To learn information such as this from an equation requires that some calculations be made.

Look again at Figure 13. The graph shows that for each date on the horizontal axis there is only one price on the vertical axis. The graph represents a function, although the exact rule for getting from date to price is not given.

When a function is defined by an equation in x and y, the **graph of the function** is the graph of the equation; that is, it is the set of points (x, y) in the xy-plane that satisfy the equation.

1 Identify the Graph of a Function

> In Words
> If any vertical line intersects a graph at more than one point, the graph is not the graph of a function.

Not every collection of points in the xy-plane represents the graph of a function. Remember, for a function, each number x in the domain has exactly one image y in the range. This means that the graph of a function cannot contain two points with the same x-coordinate and different y-coordinates. Therefore, the graph of a function must satisfy the following **vertical-line test.**

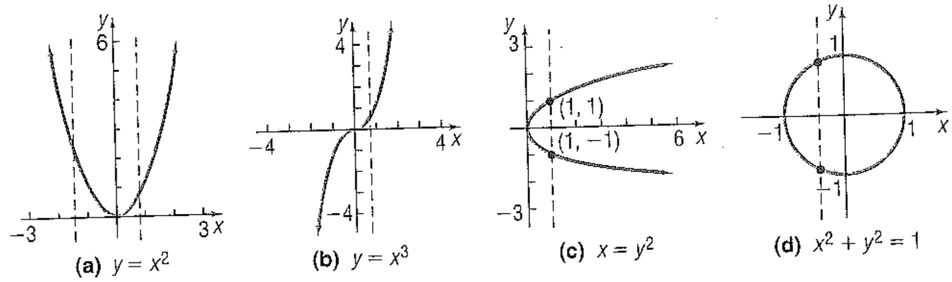
THEOREM **Vertical-Line Test**

A set of points in the xy-plane is the graph of a function if and only if every vertical line intersects the graph in at most one point.

EXAMPLE 1 **Identifying the Graph of a Function**

Which of the graphs in Figure 14 on the next page are graphs of functions?

Figure 14



Solution The graphs in Figures 14(a) and 14(b) are graphs of functions, because every vertical line intersects each graph in at most one point. The graphs in Figures 14(c) and 14(d) are not graphs of functions, because there is a vertical line that intersects each graph in more than one point. Notice in Figure 14(c) that the input 1 corresponds to two outputs, -1 and 1. This is why the graph does not represent a function.

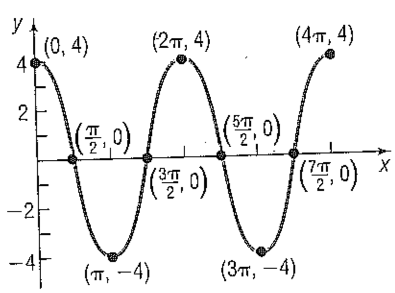
Now Work PROBLEM 15

2 Obtain Information from or about the Graph of a Function

If (x, y) is a point on the graph of a function f, then y is the value of f at x; that is, $y = f(x)$. Also if $y = f(x)$, then (x, y) is a point on the graph of f. For example, if $(-2, 7)$ is on the graph of f, then $f(-2) = 7$, and if $f(5) = 8$, then the point $(5, 8)$ is on the graph of $y = f(x)$. The next example illustrates how to obtain information about a function if its graph is given.

EXAMPLE 2 Obtaining Information from the Graph of a Function

Figure 15



Let f be the function whose graph is given in Figure 15. (The graph of f might represent the distance y that the bob of a pendulum is from its *at-rest* position at time x. Negative values of y mean that the pendulum is to the left of the at-rest position, and positive values of y mean that the pendulum is to the right of the at-rest position.)

(a) What are $f(0)$, $f\left(\frac{3\pi}{2}\right)$, and $f(3\pi)$?

(b) What is the domain of f?

(c) What is the range of f?

(d) List the intercepts. (Recall that these are the points, if any, where the graph crosses or touches the coordinate axes.)

(e) How many times does the line $y = 2$ intersect the graph?

(f) For what values of x does $f(x) = -4$?

(g) For what values of x is $f(x) > 0$?

Solution

(a) Since $(0, 4)$ is on the graph of f, the y-coordinate 4 is the value of f at the x-coordinate 0; that is, $f(0) = 4$. In a similar way, when $x = \frac{3\pi}{2}$, then $y = 0$, so $f\left(\frac{3\pi}{2}\right) = 0$. When $x = 3\pi$, then $y = -4$, so $f(3\pi) = -4$.

(b) To determine the domain of f, notice that the points on the graph of f have x-coordinates between 0 and 4π, inclusive; and for each number x between 0 and 4π, there is a point $(x, f(x))$ on the graph. The domain of f is $\{x | 0 \le x \le 4\pi\}$ or the interval $[0, 4\pi]$.

(c) The points on the graph all have y-coordinates between -4 and 4, inclusive; and for each such number y, there is at least one corresponding number x in the domain. The range of f is $\{y | -4 \le y \le 4\}$ or the interval $[-4, 4]$.

(d) The intercepts are the points

$$(0, 4), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right), \left(\frac{5\pi}{2}, 0\right), \text{ and } \left(\frac{7\pi}{2}, 0\right)$$

(e) Draw the horizontal line $y = 2$ on the graph in Figure 15. Notice that the line intersects the graph four times.

(f) Since $(\pi, -4)$ and $(3\pi, -4)$ are the only points on the graph for which $y = f(x) = -4$, we have $f(x) = -4$ when $x = \pi$ and $x = 3\pi$.

(g) To determine where $f(x) > 0$, look at Figure 15 and determine the x-values from 0 to 4π for which the y-coordinate is positive. This occurs on $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \left(\frac{7\pi}{2}, 4\pi\right]$. Using inequality notation, $f(x) > 0$ for $0 \le x < \frac{\pi}{2}$ or $\frac{3\pi}{2} < x < \frac{5\pi}{2}$ or $\frac{7\pi}{2} < x \le 4\pi$.

When the graph of a function is given, its domain may be viewed as the shadow created by the graph on the x-axis by vertical beams of light. Its range can be viewed as the shadow created by the graph on the y-axis by horizontal beams of light. Try this technique with the graph given in Figure 15.

The Zeros of a Function

If $f(r) = 0$ for a real number r, then r is called a real **zero** of f. For example, the zeros of the function $f(x) = x^2 - 4$ are -2 and 2 since $f(-2) = 0$ and $f(2) = 0$. We can identify the zeros of a function from its graph. To see how, we must remember that $y = f(x)$ means that the point (x, y) is on the graph of f. So, if r is a zero of f, then $f(r) = 0$, which means the point $(r, 0)$ is on the graph of f. That is, r is an x-intercept. We conclude that the x-intercepts of the graph of a function are also the zeros of the function.

EXAMPLE 3 **Finding the Zeros of a Function from Its Graph**

Find the zeros of the function f whose graph is shown in Figure 15.

Solution The zeros of a function are the x-intercepts of the graph of the function. Because the x-intercepts of the function shown in Figure 15 are $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, and $\frac{7\pi}{2}$, the zeros of f are $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$, and $\frac{7\pi}{2}$.

Now Work PROBLEMS 9 AND 13

EXAMPLE 4 **Obtaining Information about the Graph of a Function**

Consider the function: $f(x) = \dfrac{x+1}{x+2}$

(a) Find the domain of f.

(b) Is the point $\left(1, \frac{1}{2}\right)$ on the graph of f?

(c) If $x = 2$, what is $f(x)$? What point is on the graph of f?

(d) If $f(x) = 2$, what is x? What point is on the graph of f?

(e) What are the x-intercepts of the graph of f (if any)? What point(s) are on the graph of f?

Solution (a) The domain of f is $\{x \mid x \ne -2\}$, since $x = -2$ results in division by 0.

(b) When $x = 1$,

$$f(1) = \frac{1+1}{1+2} = \frac{2}{3}$$

The point $\left(1, \frac{2}{3}\right)$ is on the graph of f; the point $\left(1, \frac{1}{2}\right)$ is not.

(c) If $x = 2$, then

$$f(2) = \frac{2+1}{2+2} = \frac{3}{4}$$

The point $\left(2, \frac{3}{4}\right)$ is on the graph of f.

(d) If $f(x) = 2$, then

$$\begin{aligned} \frac{x+1}{x+2} &= 2 \\ x + 1 &= 2(x+2) && \text{Multiply both sides by } x + 2. \\ x + 1 &= 2x + 4 && \text{Distribute.} \\ x &= -3 && \text{Solve for } x. \end{aligned}$$

If $f(x) = 2$, then $x = -3$. The point $(-3, 2)$ is on the graph of f.

(e) The x-intercepts of the graph of f are the real solutions of the equation $f(x) = 0$ that are in the domain of f.

$$\begin{aligned} \frac{x+1}{x+2} &= 0 \\ x + 1 &= 0 && \text{Multiply both sides by } x + 2. \\ x &= -1 && \text{Solve for } x. \end{aligned}$$

The only real solution of the equation $f(x) = \frac{x+1}{x+2} = 0$ is $x = -1$, so -1 is the only x-intercept. Since $f(-1) = 0$, the point $(-1, 0)$ is on the graph of f.

Now Work PROBLEM 25

EXAMPLE 5 **Average Cost Function**

The average cost $\overline{C}$ (per computer) of manufacturing x computers per day is given by the function

$$\overline{C}(x) = 0.56x^2 - 34.39x + 1212.57 + \frac{20{,}000}{x}$$

Determine the average cost of manufacturing:

(a) 30 computers in a day (b) 40 computers in a day (c) 50 computers in a day

(d) Graph the function $\overline{C} = \overline{C}(x)$, $0 < x \le 80$.

(e) Create a TABLE with TblStart = 1 and ΔTbl = 1. Which value of x minimizes the average cost?

Solution (a) The average cost of manufacturing $x = 30$ computers is

$$\overline{C}(30) = 0.56(30)^2 - 34.39(30) + 1212.57 + \frac{20{,}000}{30} = \$1351.54$$

(b) The average cost of manufacturing $x = 40$ computers is

$$\overline{C}(40) = 0.56(40)^2 - 34.39(40) + 1212.57 + \frac{20{,}000}{40} = \$1232.97$$

(c) The average cost of manufacturing $x = 50$ computers is

$$\overline{C}(50) = 0.56(50)^2 - 34.39(50) + 1212.57 + \frac{20{,}000}{50} = \$1293.07$$

(d) See Figure 16 for the graph of $\overline{C} = \overline{C}(x)$.

(e) With the function $\overline{C} = \overline{C}(x)$ in Y_1, we create Table 2. We scroll down until we find a value of x for which Y_1 is smallest. Table 3 shows that manufacturing $x = 41$ computers minimizes the average cost at \$1231.74 per computer.

Figure 16

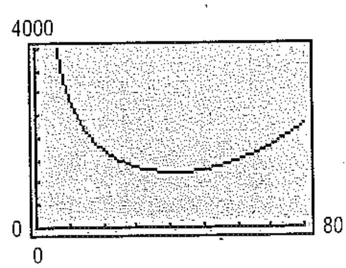


Table 2

X	Y1
1	21179
2	11146
3	7781.1
4	6084
5	5054.6
6	4359.7
7	3856.4

Y1■.56X²−34.39X...

Table 3

X	Y1
38	1240.7
39	1235.9
40	1233
41	[illegible]
42	1232.2
43	1234.4
44	1238.1

Y1=1231.744878

Now Work PROBLEM 31

SUMMARY

Graph of a Function	The collection of points (x, y) that satisfies the equation $y = f(x)$.
Vertical-Line Test	A collection of points is the graph of a function provided that every vertical line intersects the graph in at most one point.

1.2 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The intercepts of the equation $x^2 + 4y^2 = 16$ are ______________. (p. 12)

2. ***True or False*** The point $(-2, -6)$ is on the graph of the equation $x = 2y - 2$. (pp. 9–10)

Concepts and Vocabulary

3. A set of points in the xy-plane is the graph of a function if and only if every ________ line intersects the graph in at most one point.

4. If the point $(5, -3)$ is a point on the graph of f, then $f(\_\_\_) = \_\_\_\_$.

5. Find a such that the point $(-1, 2)$ is on the graph of $f(x) = ax^2 + 4$.

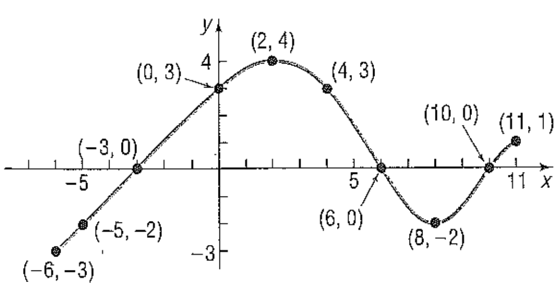
6. ***True or False*** A function can have more than one y-intercept.

7. ***True or False*** The graph of a function $y = f(x)$ always crosses the y-axis.

8. ***True or False*** The y-intercept of the graph of the function $y = f(x)$, whose domain is all real numbers, is $f(0)$.

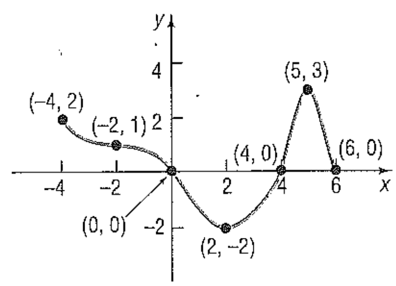
Skill Building

9. Use the given graph of the function f to answer parts (a) – (o).



(a) Find $f(0)$ and $f(-6)$.
(b) Find $f(6)$ and $f(11)$.
(c) Is $f(3)$ positive or negative?
(d) Is $f(-4)$ positive or negative?
(e) For what values of x is $f(x) = 0$?
(f) For what values of x is $f(x) > 0$?
(g) What is the domain of f?
(h) What is the range of f?
(i) What are the x-intercepts?
(j) What is the y-intercept?
(k) How many times does the line $y = \frac{1}{2}$ intersect the graph?
(l) How many times does the line $x = 5$ intersect the graph?
(m) For what values of x does $f(x) = 3$?
(n) For what values of x does $f(x) = -2$?
(o) What are the zeros of f?

10. Use the given graph of the function f to answer parts (a) – (o).

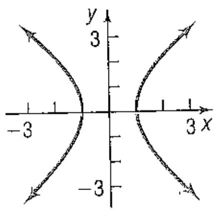


(a) Find $f(0)$ and $f(6)$.
(b) Find $f(2)$ and $f(-2)$.
(c) Is $f(3)$ positive or negative?
(d) Is $f(-1)$ positive or negative?
(e) For what values of x is $f(x) = 0$?
(f) For what values of x is $f(x) < 0$?
(g) What is the domain of f?
(h) What is the range of f?
(i) What are the x-intercepts?
(j) What is the y-intercept?
(k) How many times does the line $y = -1$ intersect the graph?
(l) How many times does the line $x = 1$ intersect the graph?
(m) For what value of x does $f(x) = 3$?
(n) For what value of x does $f(x) = -2$?
(o) What are the zeros of f?

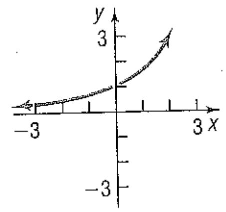
In Problems 11–22, determine whether the graph is that of a function by using the vertical-line test. If it is, use the graph to find:

(a) *The domain and range* (b) *The intercepts, if any* (c) *Any symmetry with respect to the x-axis, the y-axis, or the origin*

11.



12.



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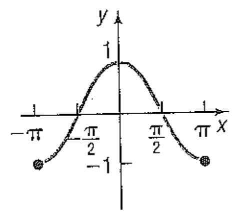

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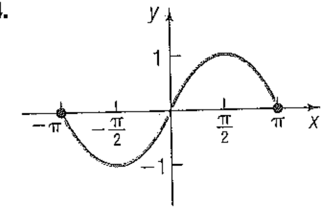

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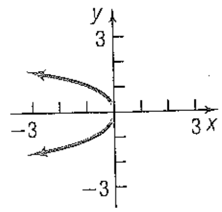

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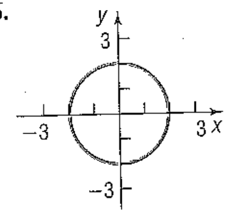
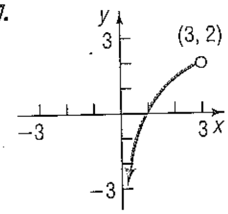


17.



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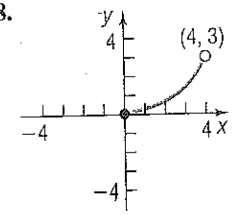

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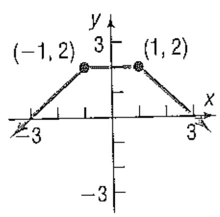

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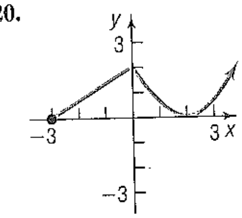
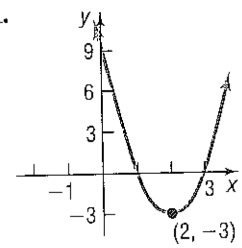
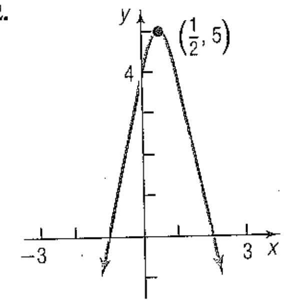


21.



22.



In Problems 23–28, answer the questions about the given function.

23. $f(x) = 2x^2 - x - 1$
(a) Is the point $(-1, 2)$ on the graph of f?
(b) If $x = -2$, what is $f(x)$? What point is on the graph of f?
(c) If $f(x) = -1$, what is x? What point(s) is (are) on the graph of f?
(d) What is the domain of f?
(e) List the x-intercepts, if any, of the graph of f.
(f) List the y-intercept, if there is one, of the graph of f.
(g) What are the zeros of f?

24. $f(x) = -3x^2 + 5x$
(a) Is the point $(-1, 2)$ on the graph of f?
(b) If $x = -2$, what is $f(x)$? What point is on the graph of f?
(c) If $f(x) = -2$, what is x? What point(s) is (are) on the graph of f?
(d) What is the domain of f?
(e) List the x-intercepts, if any, of the graph of f.
(f) List the y-intercept, if there is one, of the graph of f.
(g) What are the zeros of f?

25. $f(x) = \dfrac{x + 2}{x - 6}$
(a) Is the point $(3, 14)$ on the graph of f?
(b) If $x = 4$, what is $f(x)$? What point is on the graph of f?
(c) If $f(x) = 2$, what is x? What point(s) is (are) on the graph of f?
(d) What is the domain of f?
(e) List the x-intercepts, if any, of the graph of f.
(f) List the y-intercept, if there is one, of the graph of f.
(g) What are the zeros of f?

26. $f(x) = \dfrac{x^2 + 2}{x + 4}$
(a) Is the point $\left(1, \frac{3}{5}\right)$ on the graph of f?
(b) If $x = 0$, what is $f(x)$? What point is on the graph of f?
(c) If $f(x) = \frac{1}{2}$, what is x? What point(s) is (are) on the graph of f?
(d) What is the domain of f?
(e) List the x-intercepts, if any, of the graph of f.
(f) List the y-intercept, if there is one, of the graph of f.
(g) What are the zeros of f?

27. $f(x) = \dfrac{2x^2}{x^4 + 1}$
(a) Is the point $(-1, 1)$ on the graph of f?
(b) If $x = 2$, what is $f(x)$? What point is on the graph of f?
(c) If $f(x) = 1$, what is x? What point(s) is (are) on the graph of f?
(d) What is the domain of f?
(e) List the x-intercepts, if any, of the graph of f.
(f) List the y-intercept, if there is one, of the graph of f.
(g) What are the zeros of f?

28. $f(x) = \dfrac{2x}{x - 2}$
(a) Is the point $\left(\frac{1}{2}, -\frac{2}{3}\right)$ on the graph of f?
(b) If $x = 4$, what is $f(x)$? What point is on the graph of f?
(c) If $f(x) = 1$, what is x? What point(s) is (are) on the graph of f?
(d) What is the domain of f?
(e) List the x-intercepts, if any, of the graph of f.
(f) List the y-intercept, if there is one, of the graph of f.
(g) What are the zeros of f?

Applications and Extensions

29. Free-throw Shots According to physicist Peter Brancazio, the key to a successful foul shot in basketball lies in the arc of the shot. Brancazio determined the optimal angle of the arc from the free-throw line to be 45 degrees. The arc also depends on the velocity with which the ball is shot. If a player shoots a foul shot, releasing the ball at a 45-degree angle from a position 6 feet above the floor, then the path of the ball can be modeled by the function

$$h(x) = -\frac{44x^2}{v^2} + x + 6$$

where h is the height of the ball above the floor, x is the forward distance of the ball in front of the foul line, and v is the initial velocity with which the ball is shot in feet per second. Suppose a player shoots a ball with an initial velocity of 28 feet per second.

(a) Determine the height of the ball after it has traveled 8 feet in front of the foul line.
(b) Determine $h(12)$. What does this value represent?
(c) Find additional points, and graph the path of the basketball.
(d) The center of the hoop is 10 feet above the floor and 15 feet in front of the foul line. Will the ball go through the hoop? Why or why not? If not, with what initial velocity must the ball be shot in order for the ball to go through the hoop?

Source: The Physics of Foul Shots, Discover, *Vol. 21, No. 10, October 2000*

30. Granny Shots The last player in the NBA to use an underhand foul shot (a "granny" shot) was Hall of Fame forward Rick Barry, who retired in 1980. Barry believes that current NBA players could increase their free-throw percentage if they were to use an underhand shot. Since underhand shots are released from a lower position, the angle of the shot must be increased. If a player shoots an underhand foul shot, releasing the ball at a 70-degree angle from a position 3.5 feet above the floor, then the path of the ball can be modeled by the function $h(x) = -\frac{136x^2}{v^2} + 2.7x + 3.5$, where h is the height of the ball above the floor, x is the forward distance of the ball in front of the foul line, and v is the initial velocity with which the ball is shot in feet per second.

(a) The center of the hoop is 10 feet above the floor and 15 feet in front of the foul line. Determine the initial velocity with which the ball must be shot in order for the ball to go through the hoop.
(b) Write the function for the path of the ball using the velocity found in part (a).
(c) Determine $h(9)$. What does this value represent?
(d) Find additional points, and graph the path of the basketball.

Source: The Physics of Foul Shots, Discover, *Vol. 21, No. 10, October 2000*

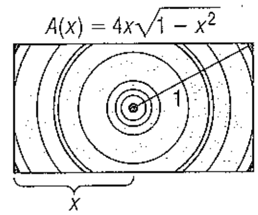
31. Motion of a Golf Ball A golf ball is hit with an initial velocity of 130 feet per second at an inclination of 45° to the horizontal. In physics, it is established that the height h of the golf ball is given by the function

$$h(x) = \frac{-32x^2}{130^2} + x$$

where x is the horizontal distance that the golf ball has traveled.

(a) Determine the height of the golf ball after it has traveled 100 feet.
(b) What is the height after it has traveled 300 feet?
(c) What is $h(500)$? Interpret this value.
(d) How far was the golf ball hit?
(e) Use a graphing utility to graph the function $h = h(x)$.
(f) Use a graphing utility to determine the distance that the ball has traveled when the height of the ball is 90 feet.
(g) Create a TABLE with TblStart = 0 and ΔTbl = 25. To the nearest 25 feet, how far does the ball travel before it reaches a maximum height? What is the maximum height?
(h) Adjust the value of ΔTbl until you determine the distance, to within 1 foot, that the ball travels before it reaches the maximum height.

32. Cross-sectional Area The cross-sectional area of a beam cut from a log with radius 1 foot is given by the function $A(x) = 4x\sqrt{1 - x^2}$, where x represents the length, in feet, of half the base of the beam. See the figure.



(a) Find the domain of A.
(b) Use a graphing utility to graph the function $A = A(x)$.
(c) Create a TABLE with TblStart = 0 and ΔTbl = 0.1. Which value of x in the domain found in part (a) maximizes the cross-sectional area? What should be the length of the base of the beam to maximize the cross-sectional area?

33. Cost of Trans-Atlantic Travel A Boeing 747 crosses the Atlantic Ocean (3000 miles) with an airspeed of 500 miles per hour. The cost C (in dollars) per passenger is given by

$$C(x) = 100 + \frac{x}{10} + \frac{36{,}000}{x}$$

where x is the ground speed (airspeed $\pm$ wind).

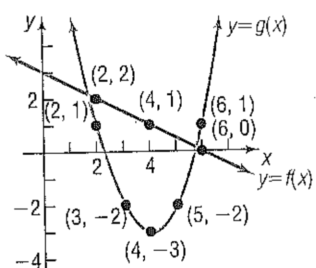
(a) What is the cost when the ground speed is 480 miles per hour; 600 miles per hour?
(b) Find the domain of C.
(c) Use a graphing utility to graph the function $C = C(x)$.
(d) Create a TABLE with TblStart $= 0$ and ΔTbl $= 50$.
(e) To the nearest 50 miles per hour, what ground speed minimizes the cost per passenger?

34. Effect of Elevation on Weight If an object weighs m pounds at sea level, then its weight W (in pounds) at a height of h miles above sea level is given approximately by

$$W(h) = m\left(\frac{4000}{4000 + h}\right)^2$$

(a) If Amy weighs 120 pounds at sea level, how much will she weigh on Pike's Peak, which is 14,110 feet above sea level?
(b) Use a graphing utility to graph the function $W = W(h)$. Use $m = 120$ pounds.
(c) Create a TABLE with TblStart $= 0$ and ΔTbl $= 0.5$ to see how the weight W varies as h changes from 0 to 5 miles.
(d) At what height will Amy weigh 119.95 pounds?
(e) Does your answer to part (d) seem reasonable? Explain.

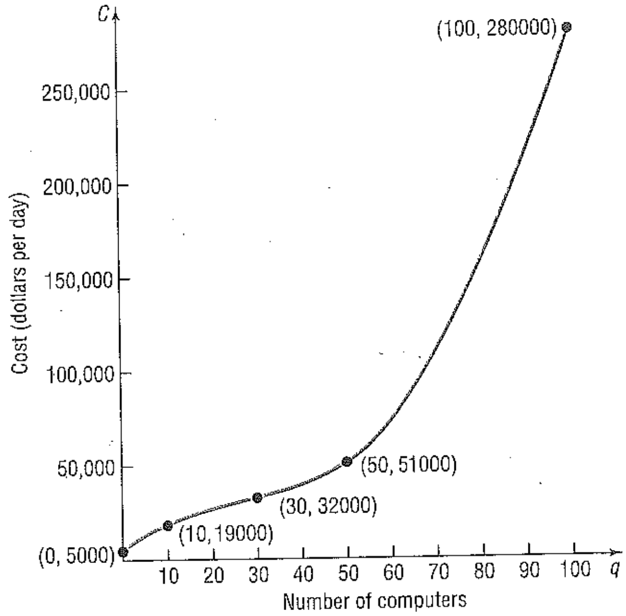
35. The graph of two functions, f and g, is illustrated. Use the graph to answer parts (a) – (f).



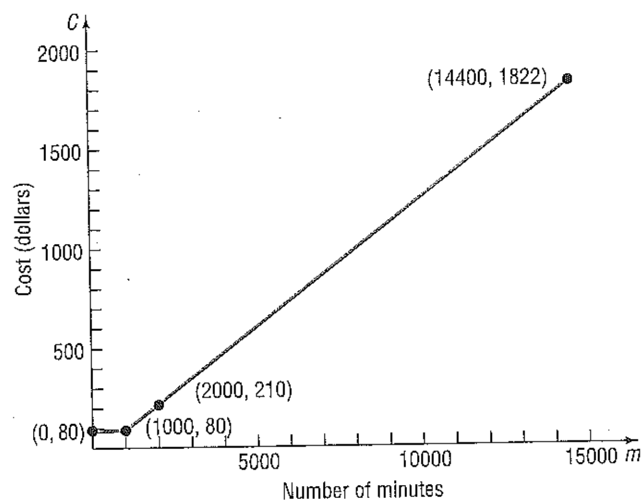
(a) $(f + g)(2)$
(b) $(f + g)(4)$
(c) $(f - g)(6)$
(d) $(g - f)(6)$
(e) $(f \cdot g)(2)$
(f) $\left(\frac{f}{g}\right)(4)$

36. Reading and Interpreting Graphs Let C be the function whose graph is given in the next column. This graph represents the cost C of manufacturing q computers in a day.

(a) Determine $C(0)$. Interpret this value.
(b) Determine $C(10)$. Interpret this value.
(c) Determine $C(50)$. Interpret this value.
(d) What is the domain of C? What does this domain imply in terms of daily production?
(e) Describe the shape of the graph.
(f) The point (30,32000) is called an *inflection point*. Describe the behavior of the graph around the inflection point.



37. Reading and Interpreting Graphs Let C be the function whose graph is given below. This graph represents the cost C of using m anytime cell phone minutes in a month for a five-person family plan.



(a) Determine $C(0)$. Interpret this value.
(b) Determine $C(1000)$. Interpret this value.
(c) Determine $C(2000)$. Interpret this value.
(d) What is the domain of C? What does this domain imply in terms of the number of anytime minutes?
(e) Describe the shape of the graph.

Discussion and Writing

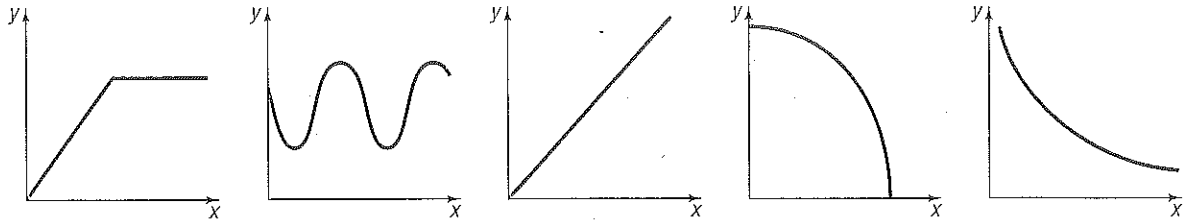
38. Describe how you would proceed to find the domain and range of a function if you were given its graph. How would your strategy change if you were given the equation defining the function instead of its graph?

39. How many x-intercepts can the graph of a function have? How many y-intercepts can the graph of a function have?

40. Is a graph that consists of a single point the graph of a function? Can you write the equation of such a function?

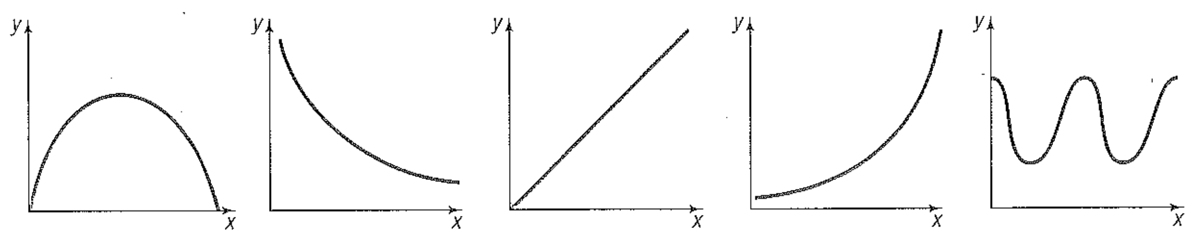
41. Match each of the following functions with the graph on the next page that best describes the situation.

(a) The cost of building a house as a function of its square footage
(b) The height of an egg dropped from a 300-foot building as a function of time
(c) The height of a human as a function of time
(d) The demand for Big Macs as a function of price
(e) The height of a child on a swing as a function of time?



(I) (II) (III) (IV) (V)

42. Match each of the following functions with the graph that best describes the situation.
    (a) The temperature of a bowl of soup as a function of time
    (b) The number of hours of daylight per day over a 2-year period
    (c) The population of Texas as a function of time
    (d) The distance traveled by a car going at a constant velocity as a function of time
    (e) The height of a golf ball hit with a 7-iron as a function of time

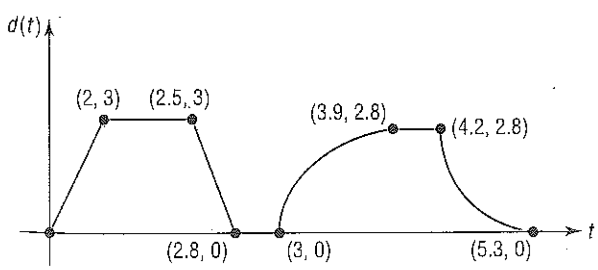


(I) (II) (III) (IV) (V)

43. Consider the following scenario: Barbara decides to take a walk. She leaves home, walks 2 blocks in 5 minutes at a constant speed, and realizes that she forgot to lock the door. So Barbara runs home in 1 minute. While at her doorstep, it takes her 1 minute to find her keys and lock the door. Barbara walks 5 blocks in 15 minutes and then decides to jog home. It takes her 7 minutes to get home. Draw a graph of Barbara's distance from home (in blocks) as a function of time.

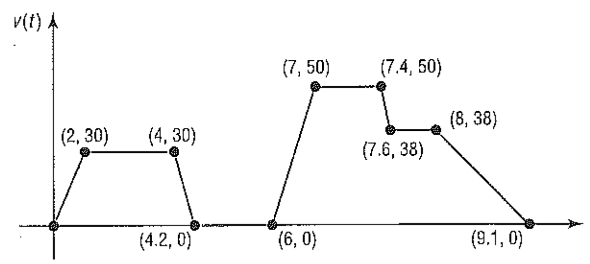
44. Consider the following scenario: Jayne enjoys riding her bicycle through the woods. At the forest preserve, she gets on her bicycle and rides up a 2000-foot incline in 10 minutes. She then travels down the incline in 3 minutes. The next 5000 feet is level terrain and she covers the distance in 20 minutes. She rests for 15 minutes. Jayne then travels 10,000 feet in 30 minutes. Draw a graph of Jayne's distance traveled (in feet) as a function of time.

45. The following sketch represents the distance d (in miles) that Kevin was from home as a function of time t (in hours). Answer the questions by referring to the graph. In parts (a) – (g), how many hours elapsed and how far was Kevin from home during this time?



(a) From $t = 0$ to $t = 2$
(b) From $t = 2$ to $t = 2.5$
(c) From $t = 2.5$ to $t = 2.8$
(d) From $t = 2.8$ to $t = 3$
(e) From $t = 3$ to $t = 3.9$
(f) From $t = 3.9$ to $t = 4.2$
(g) From $t = 4.2$ to $t = 5.3$
(h) What is the farthest distance that Kevin was from home?
(i) How many times did Kevin return home?

46. The following sketch represents the speed v (in miles per hour) of Michael's car as a function of time t (in minutes).



(a) Over what interval of time was Michael traveling fastest?
(b) Over what interval(s) of time was Michael's speed zero?
(c) What was Michael's speed between 0 and 2 minutes?
(d) What was Michael's speed between 4.2 and 6 minutes?
(e) What was Michael's speed between 7 and 7.4 minutes?
(f) When was Michael's speed constant?

47. Draw the graph of a function whose domain is $\{x \mid -3 \le x \le 8, x \ne 5\}$ and whose range is $\{y \mid -1 \le y \le 2, y \ne 0\}$. What point(s) in the rectangle $-3 \le x \le 8, -1 \le y \le 2$ cannot be on the graph? Compare your graph with those of other students. What differences do you see?

48. Is there a function whose graph is symmetric with respect to the x-axis? Explain.

49. Explain why the vertical-line test works.

'Are You Prepared?' Answers

1. $(-4, 0), (4, 0), (0, -2), (0, 2)$ 2. False

1.3 Properties of Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Intervals (Appendix A, Section A.10, pp. A81–A82)
- Intercepts (Foundations, Section 2, pp. 11–12)
- Slope of a Line (Foundations, Section 3, pp. 19–21)
- Point–Slope Form of a Line (Foundations, Section 3, p. 23)
- Symmetry (Foundations, Section 2, pp. 12–14)

Now Work the 'Are You Prepared?' problems on page 74.

OBJECTIVES
1 Determine Even and Odd Functions from a Graph (p. 66)
2 Identify Even and Odd Functions from the Equation (p. 67)
3 Use a Graph to Determine Where a Function is Increasing, Decreasing, or Constant (p. 68)
4 Use a Graph to Locate Local Maxima and Local Minima (p. 69)
5 Use a Graph to Locate the Absolute Maximum and the Absolute Minimum (p. 70)
6 Use a Graphing Utility to Approximate Local Maxima and Local Minima and to Determine Where a Function is Increasing or Decreasing (p. 71)
7 Find the Average Rate of Change of a Function (p. 72)

To obtain the graph of a function $y = f(x)$, it is often helpful to know certai properties that the function has and the impact of these properties on the way tha the graph will look.

1 Determine Even and Odd Functions from a Graph

The words *even* and *odd*, when applied to a function f, describe the symmetry thä exists for the graph of the function.

A function f is even if and only if, whenever the point (x, y) is on the grap of f, the point $(-x, y)$ is also on the graph. Using function notation, we define a even function as follows:

DEFINITION

A function f is **even** if, for every number x in its domain, the number $-x$ is also in the domain and

$$f(-x) = f(x)$$

A function f is odd if and only if, whenever the point (x, y) is on the graph of the point $(-x, -y)$ is also on the graph. Using function notation, we define an oc function as follows:

DEFINITION

A function f is **odd** if, for every number x in its domain, the number $-x$ is also in the domain and

$$f(-x) = -f(x)$$

Refer to page 13, where the tests for symmetry are listed. The following resu are then evident.

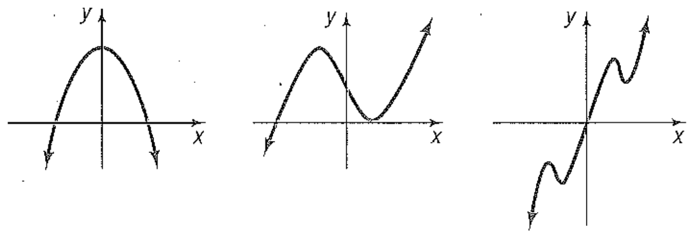
THEOREM

A function is even if and only if its graph is symmetric with respect to the y-axis. A function is odd if and only if its graph is symmetric with respect to the origin.

EXAMPLE 1 **Determining Even and Odd Functions from the Graph**

Determine whether each graph given in Figure 17 is the graph of an even function, an odd function, or a function that is neither even nor odd.

Figure 17



(a) (b) (c)

Solution

(a) The graph in Figure 17(a) is that of an even function, because the graph is symmetric with respect to the y-axis.

(b) The function whose graph is given in Figure 17(b) is neither even nor odd, because the graph is neither symmetric with respect to the y-axis nor symmetric with respect to the origin.

(c) The function whose graph is given in Figure 17(c) is odd, because its graph is symmetric with respect to the origin.

Now Work PROBLEMS 21(a), (b), AND (d)

2 Identify Even and Odd Functions from the Equation

In the next example, we use algebraic techniques to verify whether a given function is even, odd, or neither.

EXAMPLE 2 **Identifying Even and Odd Functions Algebraically**

Determine whether each of the following functions is even, odd, or neither. Then determine whether the graph is symmetric with respect to the y-axis, with respect to the origin, or neither.

(a) $f(x) = x^2 - 5$ (b) $g(x) = x^3 - 1$

(c) $h(x) = 5x^3 - x$ (d) $F(x) = |x|$

Solution

(a) To determine whether f is even, odd, or neither, replace x by $-x$ in $f(x) = x^2 - 5$. Then

$$f(-x) = (-x)^2 - 5 = x^2 - 5 = f(x)$$

Since $f(-x) = f(x)$, the function is even, and the graph of f is symmetric with respect to the y-axis.

(b) Replace x by $-x$ in $g(x) = x^3 - 1$.

$$g(-x) = (-x)^3 - 1 = -x^3 - 1$$

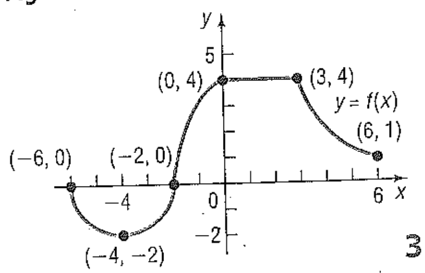
Since $g(-x) \neq g(x)$ and $g(-x) \neq -g(x) = -(x^3 - 1) = -x^3 + 1$, the function is neither even nor odd. The graph of g is not symmetric with respect to the y-axis, nor is it symmetric with respect to the origin.

(c) Replace x by $-x$ in $h(x) = 5x^3 - x$.

$$h(-x) = 5(-x)^3 - (-x) = -5x^3 + x = -(5x^3 - x) = -h(x)$$

Since $h(-x) = -h(x)$, h is an odd function, and the graph of h is symmetric with respect to the origin.

Figure 18



(d) Replace x by $-x$ in $F(x) = |x|$. Then

$$F(-x) = |-x| = |-1| \cdot |x| = |x| = F(x)$$

Since $F(-x) = F(x)$, F is an even function, and the graph of F is symmetric with respect to the y-axis.

Now Work PROBLEM 33

3 Use a Graph to Determine Where a Function Is Increasing, Decreasing, or Constant

Consider the graph given in Figure 18. Look from left to right along the graph of the function, and note that parts of the graph are going up, parts are going down, and parts are horizontal. In such cases, the function is described as *increasing*, *decreasing*, or *constant*, respectively.

EXAMPLE 3 Determining Where a Function Is Increasing, Decreasing, or Constant from Its Graph

Determine the values of x for which the function in Figure 18 is increasing. Where is it decreasing? Where is it constant?

Solution To determine where a function is increasing, where it is decreasing, and where it is constant, we use strict inequalities involving the independent variable x, or we use open intervals* of x-coordinates. The function whose graph is given in Figure 18 is increasing on the open interval $(-4, 0)$, or for $-4 < x < 0$. The function is decreasing on the open intervals $(-6, -4)$ and $(3, 6)$, or for $-6 < x < -4$ and $3 < x < 6$. The function is constant on the open interval $(0, 3)$, or for $0 < x < 3$.

WARNING Describe the behavior of a graph in terms of its x-values. Do not say the graph in Figure 18 is increasing from the point $(-4, -2)$ to the point $(0, 4)$. Rather, say it is increasing on the interval $(-4, 0)$.

More precise definitions follow:

DEFINITIONS

A function f is **increasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

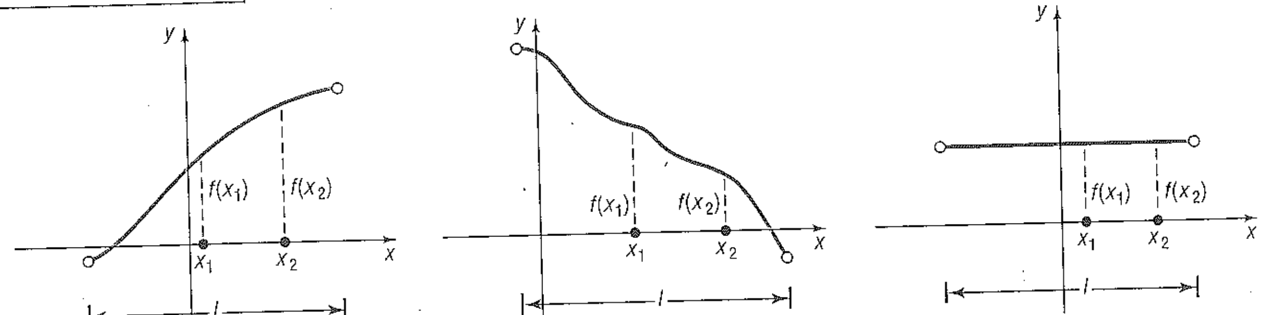
A function f is **decreasing** on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

A function f is **constant** on an open interval I if, for all choices of x in I, the values of $f(x)$ are equal.

In Words
If a function is decreasing, then, as the values of x get bigger, the values of the function get smaller. If a function is increasing, then, as the values of x get bigger, the values of the function also get bigger. If a function is constant, then, as the values of x get bigger, the values of the function remain unchanged.

Figure 19 illustrates the definitions. Graphs are read like a book—from left to right. Thus, the graph of an increasing function goes up from left to right, the graph of a decreasing function goes down from left to right, and the graph of a constant function remains at a fixed height.

Figure 19



(a) For $x_1 < x_2$ in I, $f(x_1) < f(x_2)$; f is increasing on I.

(b) For $x_1 < x_2$ in I, $f(x_1) > f(x_2)$; f is decreasing on I.

(c) For all x in I, the values of f are equal; f is constant on I.

Now Work PROBLEMS 11, 13, 15, AND 21(c)

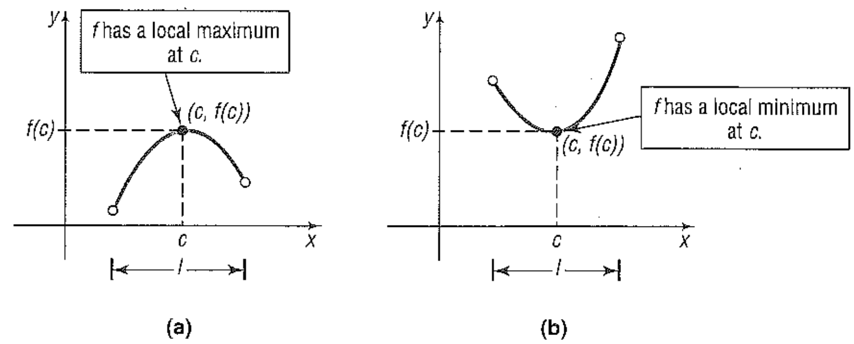
*The open interval (a, b) consists of all real numbers x for which $a < x < b$.

4 Use a Graph to Locate Local Maxima and Local Minima

Suppose f is a function defined on an open interval I containing c. If the value of f at c is greater than or equal to the values of f on I, then f has a *local maximum* at c.* See Figure 20(a).

If the value of f at c is less than or equal to the values of f on I, then f has a *local minimum* at c. See Figure 20(b).

Figure 20



DEFINITIONS

A function f has a **local maximum** at c if there is an open interval I containing c so that, for all x in I, $f(x) \leq f(c)$. We call $f(c)$ a **local maximum value of f.**

A function f has a **local minimum** at c if there is an open interval I containing c so that, for all x in I, $f(x) \geq f(c)$. We call $f(c)$ a **local minimum value of f.**

If f has a local maximum at c, then the value of f at c is greater than or equal to the values of f near c. If f has a local minimum at c, then the value of f at c is less than or equal to the values of f near c. The word *local* is used to suggest that it is only near c, not necessarily over the entire domain, that the value $f(c)$ has these properties.

EXAMPLE 4 **Finding Local Maxima and Local Minima from the Graph of a Function and Determining Where the Function Is Increasing, Decreasing, or Constant**

Figure 21

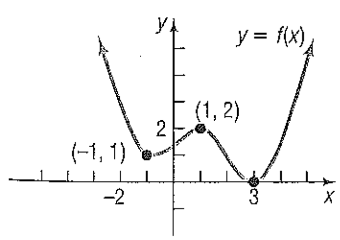


Figure 21 shows the graph of a function f.

(a) At what value(s) of x, if any, does f have a local maximum? List the local maximum values.

(b) At what value(s) of x, if any, does f have a local minimum? List the local minimum values.

(c) Find the intervals on which f is increasing. Find the intervals on which f is decreasing.

Solution The domain of f is the set of real numbers.

(a) f has a local maximum at 1, since for all x close to 1, we have $f(x) \leq f(1)$. The local maximum value is $f(1) = 2$.

(b) f has local minima at -1 and at 3. The local minimum values are $f(-1) = 1$ and $f(3) = 0$.

(c) The function whose graph is given in Figure 21 is increasing for all values of x between -1 and 1 and for all values of x greater than 3. That is, the function is increasing on the intervals $(-1, 1)$ and $(3, \infty)$, or for $-1 < x < 1$ and $x > 3$. The function is decreasing for all values of x less than -1 and for all values of x

WARNING The y-value is the local maximum value or local minimum value, and it occurs at some x-value. For example, in Figure 21, we say f has a local maximum at 1 and the local maximum value is 2.

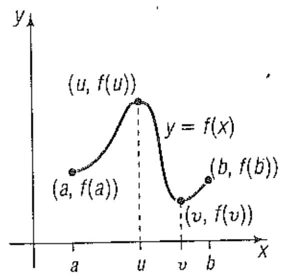
*Some texts use the term *relative* instead of *local*.

between 1 and 3. That is, the function is decreasing on the intervals $(-\infty, -1)$ and $(1, 3)$, or for $x < -1$ and $1 < x < 3$.

Now Work PROBLEMS 17 AND 19

5 Use a Graph to Locate the Absolute Maximum and the Absolute Minimum

Figure 22



domain: $[a, b]$
for all x in $[a, b]$, $f(x) \leq f(u)$
for all x in $[a, b]$, $f(x) \geq f(v)$
absolute maximum: $f(u)$
absolute minimum: $f(v)$

Look at the graph of the function f given in Figure 22. The domain of f is the closed interval $[a, b]$. Also, the largest value of f is $f(u)$ and the smallest value of f is $f(v)$. These are called, respectively, the *absolute maximum* and the *absolute minimum* of f on $[a, b]$.

DEFINITION Let f denote a function defined on some interval I. If there is a number u in I for which $f(x) \leq f(u)$ for all x in I, then $f(u)$ is the **absolute maximum of f** on I, and we say **the absolute maximum of f occurs at u.**

If there is a number v in I for which $f(x) \geq f(v)$ for all x in I, then $f(v)$ is the **absolute minimum of f** on I, and we say **the absolute minimum of f occurs at v.**

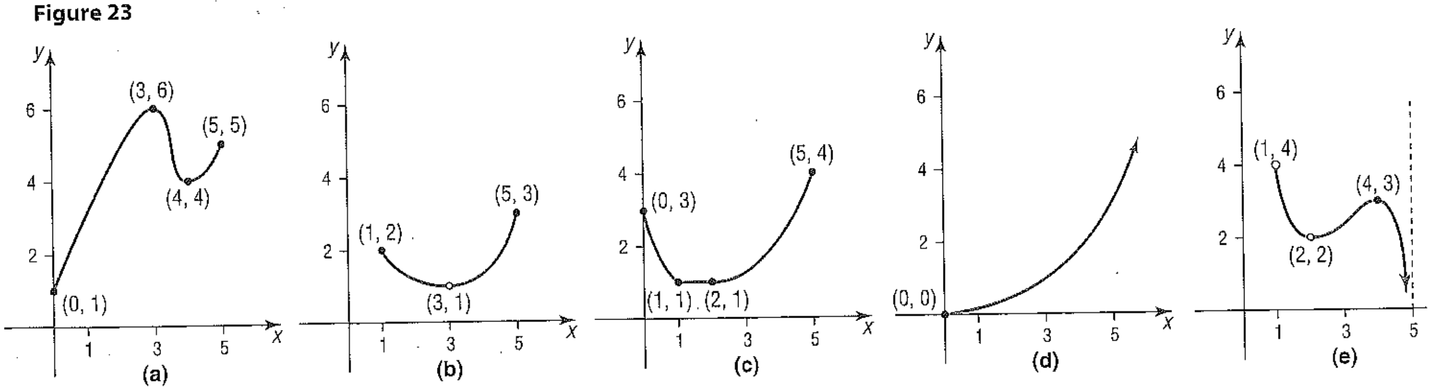
The absolute maximum and absolute minimum of a function f are sometimes called the **extreme values** of f on I.

The absolute maximum or absolute minimum of a function f may not exist. Let's look at some examples.

EXAMPLE 5 Finding the Absolute Maximum and the Absolute Minimum from the Graph of a Function

For each graph of a function $y = f(x)$ in Figure 23, find the absolute maximum and the absolute minimum, if they exist. Also, find any local maxima or local minima.

Figure 23



Solution (a) The function f whose graph is given in Figure 23(a) has the closed interval $[0, 5]$ as its domain. The largest value of f is $f(3) = 6$, the absolute maximum. The smallest value of f is $f(0) = 1$, the absolute minimum. The function has a local maximum of 6 at $x = 3$ and a local minimum of 4 at $x = 4$.

(b) The function f whose graph is given in Figure 23(b) has the domain $\{x | 1 \leq x \leq 5, x \neq 3\}$. Note that we exclude 3 from the domain because of the "hole" at $(3, 1)$. The largest value of f on its domain is $f(5) = 3$, the absolute maximum. There is no absolute minimum. Do you see why? As you trace the graph, getting closer to the point $(3, 1)$, there is no single smallest value. [As soon as you claim a smallest value, we can trace closer to $(3, 1)$ and get a smaller value!] The function has no local maxima or minima.

WARNING A function may have an absolute maximum or an absolute minimum at an endpoint, but not a local maximum or a local minimum. Why? Local maxima and local minima are found over some open interval I, and this interval cannot be created around an endpoint.

(c) The function f whose graph is given in Figure 23(c) has the interval $[0, 5]$ as its domain. The absolute maximum of f is $f(5) = 4$. The absolute minimum is 1. Notice that the absolute minimum 1 occurs at any number in the interval $[1, 2]$. The function has a local minimum of 1 at every x in the interval $[1, 2]$, but it has no local maxima.

(d) The graph of the function f given in Figure 23(d) has the interval $[0, \infty)$ as its domain. The function has no absolute maximum; the absolute minimum is $f(0) = 0$. The function has no local maxima or local minima.

(e) The graph of the function f in Figure 23(e) has the domain $\{x|1 < x < 5, x \neq 2\}$. The function f has no absolute maximum and no absolute minimum. Do you see why? The function has a local maximum of 3 at $x = 4$, but no local minima. ●

In calculus, there is a theorem with conditions that guarantee a function will have an absolute maximum and an absolute minimum.

THEOREM

Extreme Value Theorem

If f is a continuous function* whose domain is a closed interval $[a, b]$, then f has an absolute maximum and an absolute minimum on $[a, b]$.

The absolute maximum (minimum) can be found by selecting the largest (smallest) value of f from the following list:

1. The value of f at any local maxima and local minima of f in (a, b).
2. The value of f at each endpoint of $[a, b]$—that is, $f(a)$ and $f(b)$.
3. The value of f on any interval in $[a, b]$ on which f is constant.

For example, the graph of the function f given in Figure 23(a) is continuous on the closed interval $[0, 5]$. The Extreme Value Theorem guarantees that f has extreme values on $[0, 5]$. To find them, we list

1. The value of f at the local extrema: $f(3) = 6, f(4) = 4$
2. The value of f at the endpoints: $f(0) = 1, f(5) = 5$

The largest of these, 6, is the absolute maximum; the smallest of these, 1, is the absolute minimum.

Now Work PROBLEM 45

6 Use a Graphing Utility to Approximate Local Maxima and Local Minima and to Determine Where a Function Is Increasing or Decreasing

To locate the exact value at which a function f has a local maximum or a local minimum usually requires calculus. However, a graphing utility may be used to approximate these values by using the MAXIMUM and MINIMUM features.

EXAMPLE 6

Using a Graphing Utility to Approximate Local Maxima and Minima and to Determine Where a Function Is Increasing or Decreasing

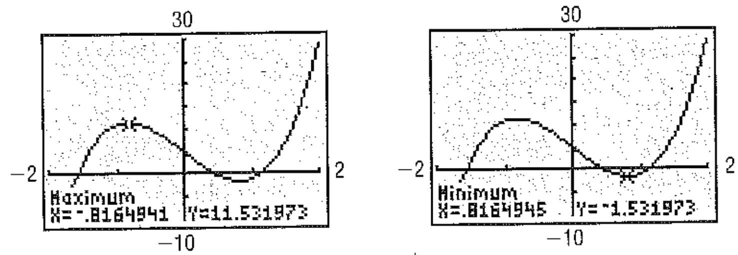
(a) Use a graphing utility to graph $f(x) = 6x^3 - 12x + 5$ for $-2 < x < 2$. Approximate where f has a local maximum and where f has a local minimum.

(b) Determine where f is increasing and where it is decreasing.

*Although a precise definition requires calculus, we'll agree for now that a continuous function is one whose graph has no gaps or holes and can be traced without lifting the pencil from the paper.

Solution (a) Graphing utilities have a feature that finds the maximum or minimum point of a graph within a given interval. Graph the function f for $-2 < x < 2$. The MAXIMUM and MINIMUM commands require us to first determine the open interval I. The graphing utility will then approximate the maximum or minimum value in the interval. Using MAXIMUM, we find that the local maximum is 11.53 and that it occurs at $x = -0.82$, rounded to two decimal places. See Figure 24(a). Using MINIMUM, we find that the local minimum is -1.53 and that it occurs at $x = 0.82$, rounded to two decimal places. See Figure 24(b).

Figure 24



(a) (b)

(b) Looking at Figures 24(a) and (b), we see that the graph of f is increasing from $x = -2$ to $x = -0.82$ and from $x = 0.82$ to $x = 2$, so f is increasing on the intervals $(-2, -0.82)$ and $(0.82, 2)$, or for $-2 < x < -0.82$ and $0.82 < x < 2$. The graph is decreasing from $x = -0.82$ to $x = 0.82$, so f is decreasing on the interval $(-0.82, 0.82)$, or for $-0.82 < x < 0.82$.

Now Work PROBLEM 53

7 Find the Average Rate of Change of a Function

In Foundations, Section 3, we said that the slope of a line could be interpreted as the average rate of change. To find the average rate of change of a function between any two points on its graph, calculate the slope of the line containing the two points.

DEFINITION

If a and b, $a \neq b$, are in the domain of a function $y = f(x)$, the **average rate of change of f from a to b** is defined as

$$\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} \qquad a \neq b \qquad (1)$$

In Words
The symbol Δ is the Greek capital letter delta and is read "change in."

The symbol Δy in equation (1) is the "change in y," and Δx is the "change in x." The average rate of change of f is the change in y divided by the change in x.

EXAMPLE 7 **Finding the Average Rate of Change**

Find the average rate of change of $f(x) = 3x^2$:

(a) From 1 to 3 (b) From 1 to 5 (c) From 1 to 7

Solution (a) The average rate of change of $f(x) = 3x^2$ from 1 to 3 is

$$\frac{\Delta y}{\Delta x} = \frac{f(3) - f(1)}{3 - 1} = \frac{27 - 3}{3 - 1} = \frac{24}{2} = 12$$

(b) The average rate of change of $f(x) = 3x^2$ from 1 to 5 is

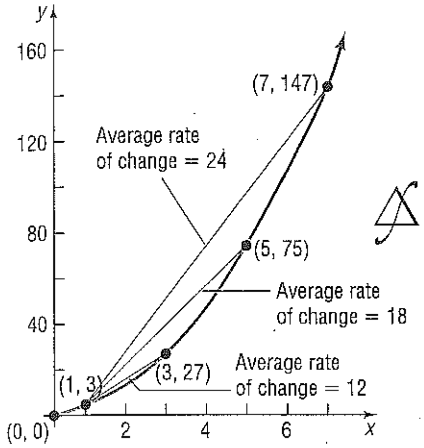
$$\frac{\Delta y}{\Delta x} = \frac{f(5) - f(1)}{5 - 1} = \frac{75 - 3}{5 - 1} = \frac{72}{4} = 18$$

(c) The average rate of change of $f(x) = 3x^2$ from 1 to 7 is

$$\frac{\Delta y}{\Delta x} = \frac{f(7) - f(1)}{7 - 1} = \frac{147 - 3}{7 - 1} = \frac{144}{6} = 24$$

See Figure 25 for a graph of $f(x) = 3x^2$. The function f is increasing for $x > 0$. The fact that the average rate of change is positive for any $x_1, x_2, x_1 \neq x_2$, in the interval $(1, 7)$ indicates that the graph is increasing on $1 < x < 7$. Further, the average rate of change is consistently getting larger for $1 < x < 7$, which indicates that the graph is increasing at an increasing rate.

Figure 25



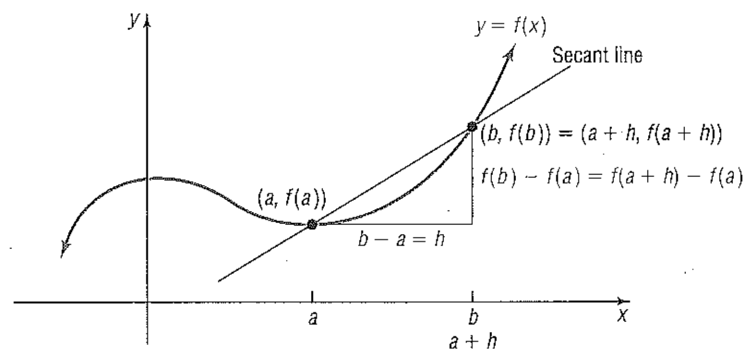
Now Work PROBLEM 61

The Secant Line

The average rate of change of a function has an important geometric interpretation. Look at the graph of $y = f(x)$ in Figure 26. Two points are labeled on the graph: $(a, f(a))$ and $(b, f(b))$. The line containing these two points is called the **secant line;** its slope is

$$m_{\text{sec}} = \frac{f(b) - f(a)}{b - a} = \frac{f(a + h) - f(a)}{h}$$

Figure 26



THEOREM

Slope of the Secant Line

The average rate of change of a function f from a to b equals the slope of the secant line containing the two points $(a, f(a))$ and $(b, f(b))$ on its graph.

EXAMPLE 8 **Finding the Equation of a Secant Line**

Suppose that $g(x) = 3x^2 - 2x + 3$.

(a) Find the average rate of change of g from -2 to 1.

(b) Find an equation of the secant line containing $(-2, g(-2))$ and $(1, g(1))$.

(c) Using a graphing utility, draw the graph of g and the secant line obtained in part (b) on the same screen.

Solution (a) The average rate of change of $g(x) = 3x^2 - 2x + 3$ from -2 to 1 is

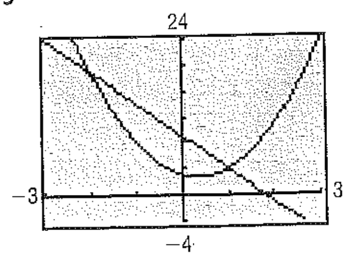
$$\text{Average rate of change} = \frac{g(1) - g(-2)}{1 - (-2)}$$

$$= \frac{4 - 19}{3} \qquad g(1) = 3(1)^2 - 2(1) + 3 = 4,\quad g(-2) = 3(-2)^2 - 2(-2) + 3 = 19$$

$$= -\frac{15}{3} = -5$$

(b) The slope of the secant line containing $(-2, g(-2)) = (-2, 19)$ and $(1, g(1)) = (1, 4)$ is $m_{\text{sec}} = -5$. Use the point–slope form to find an equation of the secant line.

Figure 27



$$y - y_1 = m_{sec}(x - x_1) \quad \text{Point–slope form of the secant line}$$
$$y - 19 = -5(x - (-2)) \quad x_1 = -2, y_1 = g(-2) = 19, m_{sec} = -5$$
$$y - 19 = -5x - 10 \quad \text{Distribute.}$$
$$y = -5x + 9 \quad \text{Slope–intercept form of the secant line}$$

(c) Figure 27 shows the graph of g along with the secant line $y = -5x + 9$.

Now Work PROBLEM 67

1.3 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

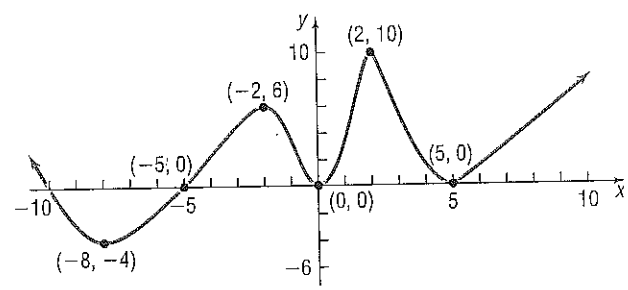
1. The interval $(2, 5)$ can be written as the inequality ________. (pp. A81–A82)
2. The slope of the line containing the points $(-2, 3)$ and $(3, 8)$ is ____. (pp. 19–21)
3. Test the equation $y = 5x^2 - 1$ for symmetry with respect to the x-axis, the y-axis, and the origin. (pp. 12–14)
4. Write the point–slope form of the line with slope 5 containing the point $(3, -2)$. (p. 23)
5. The intercepts of the equation $y = x^2 - 9$ are ____________. (pp. 11–12)

Concepts and Vocabulary

6. A function f is ________ on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.
7. A(n) ______ function f is one for which $f(-x) = f(x)$ for every x in the domain of f; a(n) ______ function f is one for which $f(-x) = -f(x)$ for every x in the domain of f.
8. ***True or False*** A function f is decreasing on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) > f(x_2)$.
9. ***True or False*** A function f has a local maximum at c if there is an open interval I containing c so that for all x in I, $f(x) \le f(c)$.
10. ***True or False*** Even functions have graphs that are symmetric with respect to the origin.

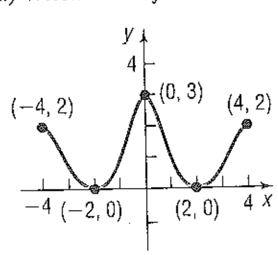
Skill Building

In Problems 11–20, use the graph of the function f given.



11. Is f increasing on the interval $(-8, -2)$?
12. Is f decreasing on the interval $(-8, -4)$?
13. Is f increasing on the interval $(2, 10)$?
14. Is f decreasing on the interval $(2, 5)$?
15. List the interval(s) on which f is increasing.
16. List the interval(s) on which f is decreasing.
17. Is there a local maximum value at 2? If yes, what is it?
18. Is there a local maximum value at 5? If yes, what is it?
19. List the number(s) at which f has a local maximum. What are the local maximum values?
20. List the number(s) at which f has a local minimum. What are the local minimum values?

In Problems 21–28, the graph of a function is given. Use the graph to find:
(a) The intercepts, if any
(b) The domain and range
(c) The intervals on which the function is increasing, decreasing, or constant
(d) Whether the function is even, odd, or neither

21.


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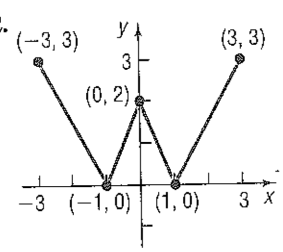

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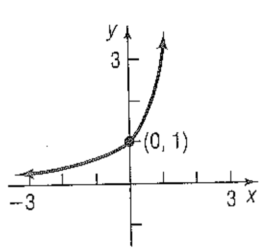

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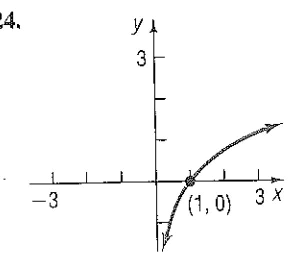

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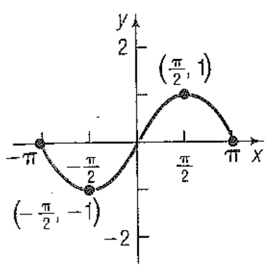

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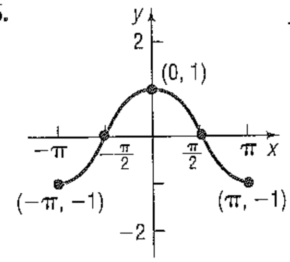

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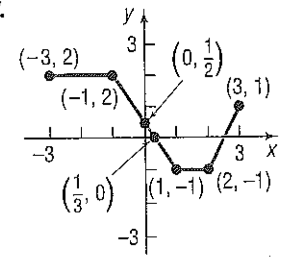

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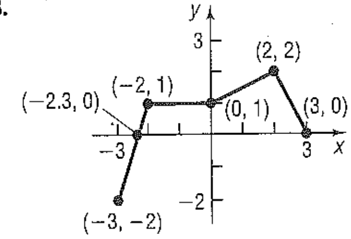
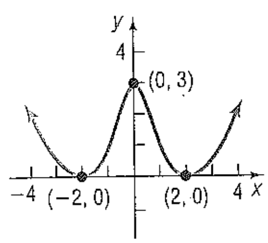


In Problems 29–32, the graph of a function f is given. Use the graph to find:
(a) The numbers, if any, at which f has a local maximum value. What are the local maximum values?
(b) The numbers, if any, at which f has a local minimum value. What are the local minimum values?

29.


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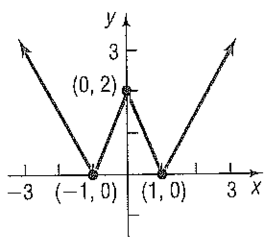

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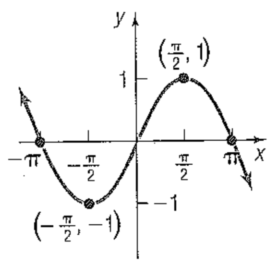

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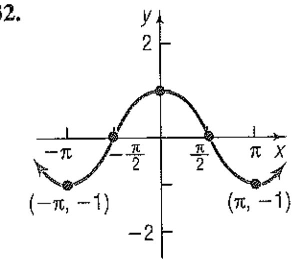
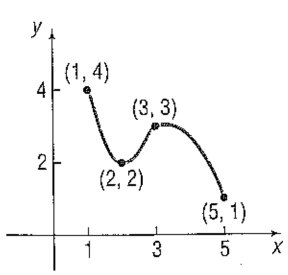


In Problems 33–44, determine algebraically whether each function is even, odd, or neither.

33. $f(x) = 4x^3$
34. $f(x) = 2x^4 - x^2$
35. $g(x) = -3x^2 - 5$
36. $h(x) = 3x^3 + 5$
37. $F(x) = \sqrt[3]{x}$
38. $G(x) = \sqrt{x}$
39. $f(x) = x + |x|$
40. $f(x) = \sqrt[3]{2x^2 + 1}$
41. $g(x) = \dfrac{x^2 + 3}{x^2 - 1}$
42. $h(x) = \dfrac{x}{x^2 - 1}$
43. $h(x) = \dfrac{-x^3}{3x^2 - 9}$
44. $F(x) = \dfrac{2x}{|x|}$

In Problems 45–52, for each graph of a function $y = f(x)$, find the absolute maximum and the absolute minimum, if they exist. Identify any local maxima or local minima.

45.


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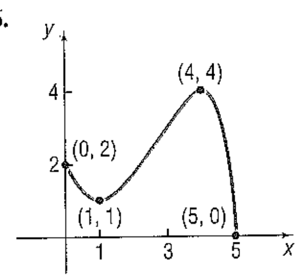

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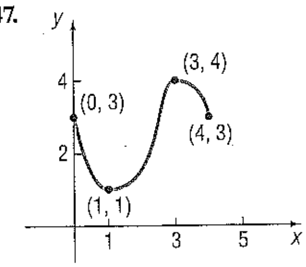

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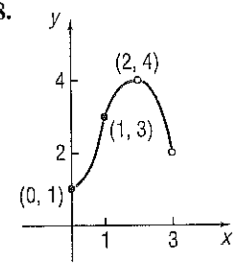
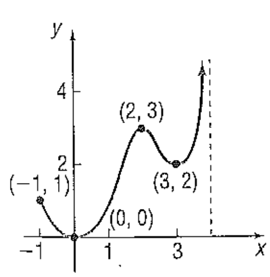


49.


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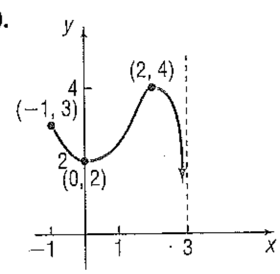
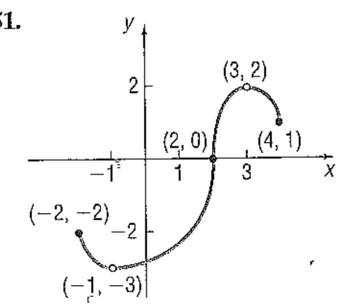
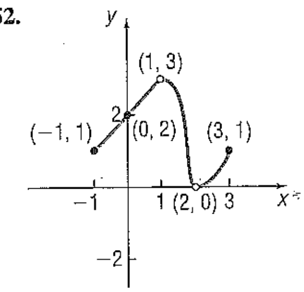


51.


52.


In Problems 53–60, use a graphing utility to graph each function over the indicated interval and approximate any local maximum values and local minimum values. Determine where the function is increasing and where it is decreasing. Round answers to two decimal places.

53. $f(x) = x^3 - 3x + 2 \quad (-2, 2)$
54. $f(x) = x^3 - 3x^2 + 5 \quad (-1, 3)$
55. $f(x) = x^5 - x^3 \quad (-2, 2)$
56. $f(x) = x^4 - x^2 \quad (-2, 2)$
57. $f(x) = -0.2x^3 - 0.6x^2 + 4x - 6 \quad (-6, 4)$
58. $f(x) = -0.4x^3 + 0.6x^2 + 3x - 2 \quad (-4, 5)$
59. $f(x) = 0.25x^4 + 0.3x^3 - 0.9x^2 + 3 \quad (-3, 2)$
60. $f(x) = -0.4x^4 - 0.5x^3 + 0.8x^2 - 2 \quad (-3, 2)$

61. Find the average rate of change of $f(x) = -2x^2 + 4$.
  (a) From 0 to 2
  (b) From 1 to 3
  (c) From 1 to 4

62. Find the average rate of change of $f(x) = -x^3 + 1$.
  (a) From 0 to 2
  (b) From 1 to 3
  (c) From −1 to 1

63. Find the average rate of change of $g(x) = x^3 - 2x + 1$.
 (a) From -3 to -2
 (b) From -1 to 1
 (c) From 1 to 3

64. Find the average rate of change of $h(x) = x^2 - 2x + 3$.
 (a) From -1 to 1
 (b) From 0 to 2
 (c) From 2 to 5

65. $f(x) = 5x - 2$
 (a) Find the average rate of change from 1 to 3.
 (b) Find an equation of the secant line containing $(1, f(1))$ and $(3, f(3))$.

66. $f(x) = -4x + 1$
 (a) Find the average rate of change from 2 to 5.
 (b) Find an equation of the secant line containing $(2, f(2))$ and $(5, f(5))$.

67. $g(x) = x^2 - 2$
 (a) Find the average rate of change from -2 to 1.
 (b) Find an equation of the secant line containing $(-2, g(-2))$ and $(1, g(1))$.

68. $g(x) = x^2 + 1$
 (a) Find the average rate of change from -1 to 2.
 (b) Find an equation of the secant line containing $(-1, g(-1)$ and $(2, g(2))$.

69. $h(x) = x^2 - 2x$
 (a) Find the average rate of change from 2 to 4.
 (b) Find an equation of the secant line containing $(2, h(2)$ and $(4, h(4))$.

70. $h(x) = -2x^2 + x$
 (a) Find the average rate of change from 0 to 3.
 (b) Find an equation of the secant line containing $(0, h(0)$ and $(3, h(3))$.

Mixed Practice

71. $g(x) = x^3 - 27x$
 (a) Determine whether g is even, odd, or neither.
 (b) There is a local minimum value of -54 at 3. Determine a local maximum value.

72. $f(x) = -x^3 + 12x$
 (a) Determine whether f is even, odd, or neither.
 (b) There is a local maximum value of 16 at 2. Determine a local minimum value.

73. $F(x) = -x^4 + 8x^2 + 8$
 (a) Determine whether F is even, odd, or neither.
 (b) There is a local maximum value of 24 at $x = 2$. Determine a second local maximum value.
 (c) Suppose the area under the graph of F between $x = 0$ and $x = 3$ that is bounded below by the x-axis is 47.4 square units. Using the result from part (a), determine the area under the graph of F between $x = -3$ and $x = 0$ bounded below by the x-axis.

74. $G(x) = -x^4 + 32x^2 + 144$
 (a) Determine whether G is even, odd, or neither.
 (b) There is a local maximum value of 400 at $x = 4$. Determine a second local maximum value.
 (c) Suppose the area under the graph of G between $x = ($ and $x = 6$ that is bounded below by the x-axis is 1612.8 square units. Using the result from part (a), determine the area under the graph of G between $x = -6$ and $x = 0$ bounded below by the x-axis.

Applications and Extensions

75. **Minimum Average Cost** The average cost per hour in dollars, $\overline{C}$, of producing x riding lawn mowers can be modeled by the function

$$\overline{C}(x) = 0.3x^2 + 21x - 251 + \frac{2500}{x}$$

 (a) Use a graphing utility to graph $\overline{C} = \overline{C}(x)$.
 (b) Determine the number of riding lawn mowers to produce in order to minimize average cost.
 (c) What is the minimum average cost?

76. **Medicine Concentration** The concentration C of a medication in the bloodstream t hours after being administered is modeled by the function

$$C(t) = -0.002t^4 + 0.039t^3 - 0.285t^2 + 0.766t + 0.085$$

 (a) After how many hours will the concentration be highest?
 (b) A woman nursing a child must wait until the concentration is below 0.5 before she can feed her or him. After taking the medication, how long must she wait before feeding her child?

77. **National Debt** The size of the total debt owed by the United States federal government continues to grow. In fact, according to the Department of the Treasury, the debt per person living in the United States is approximately \$45,000 (or over \$300,000 per U.S. household). The following data represent the U.S. debt for the years 2001–2012. Since tl debt D depends on the year y, and each input correspon to exactly one output, the debt is a function of the year. Th $D(y)$, represents the debt for each year y.

Year	Debt (billions of dollars)	Year	Debt (billions of dollars)
2001	5807	2007	9008
2002	6228	2008	10,025
2003	6783	2009	11,910
2004	7379	2010	13,562
2005	7933	2011	14,790
2006	8507	2012	16,066

Source: www.treasurydirect.gov

 (a) Plot the points (2001, 5807), (2002, 6228), and so on i Cartesian plane.
 (b) Draw a line segment from the point (2001, 5807) (2006, 8507). What does the slope of this line segm represent?
 (c) Find the average rate of change of the debt from 200 2004.
 (d) Find the average rate of change of the debt from 200 2008.

(e) Find the average rate of change of the debt from 2010 to 2012.

(f) What is happening to the average rate of change as time passes?

78. ***E.coli* Growth** A strain of *E.coli* Beu 397-recA441 is placed into a nutrient broth at 30° Celsius and allowed to grow. The data shown in the table are collected. The population is measured in grams and the time in hours. Since population P depends on time t, and each input corresponds to exactly one output, we can say that population is a function of time. Thus $P(t)$ represents the population at time t.

Time (hours), t	Population (grams), P
0	0.09
2.5	0.18
3.5	0.26
4.5	0.35
6	0.50

(a) Plot the points (0, 0.09), (2.5, 0.18), and so on in a Cartesian plane.

(b) Draw a line segment from the point (0, 0.09) to (2.5, 0.18). What does the slope of this line segment represent?

(c) Find the average rate of change of the population from 0 to 2.5 hours.

(d) Find the average rate of change of the population from 4.5 to 6 hours.

(e) What is happening to the average rate of change as time passes?

79. For the function $f(x) = x^2$, compute each average rate of change:

(a) From 0 to 1

(b) From 0 to 0.5

(c) From 0 to 0.1

(d) From 0 to 0.01

(e) From 0 to 0.001

(f) Use a graphing utility to graph each of the secant lines along with f.

(g) What do you think is happening to the secant lines?

(h) What is happening to the slopes of the secant lines? Is there some number that they are getting closer to? What is that number?

80. For the function $f(x) = x^2$, compute each average rate of change:

(a) From 1 to 2

(b) From 1 to 1.5

(c) From 1 to 1.1

(d) From 1 to 1.01

(e) From 1 to 1.001

(f) Use a graphing utility to graph each of the secant lines along with f.

(g) What do you think is happening to the secant lines?

(h) What is happening to the slopes of the secant lines? Is there some number that they are getting closer to? What is that number?

Problems 81–88 require the following discussion of a secant line. The slope of the secant line containing the two points $(x, f(x))$ and $(x + h, f(x + h))$ on the graph of a function $y = f(x)$ may be given as

$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h} \qquad h \neq 0$$

In calculus, this expression is called the **difference quotient of *f*.**

(a) Express the slope of the secant line of each function in terms of x and h. Be sure to simplify your answer.

(b) Find m_{sec} for $h = 0.5, 0.1,$ and 0.01 at $x = 1$. What value does m_{sec} approach as h approaches 0?

(c) Find the equation for the secant line at $x = 1$ with $h = 0.01$.

(d) Use a graphing utility to graph f and the secant line found in part (c) on the same viewing window.

81. $f(x) = 2x + 5$

82. $f(x) = -3x + 2$

83. $f(x) = x^2 + 2x$

84. $f(x) = 2x^2 + x$

85. $f(x) = 2x^2 - 3x + 1$

86. $f(x) = -x^2 + 3x - 2$

87. $f(x) = \dfrac{1}{x}$

88. $f(x) = \dfrac{1}{x^2}$

Discussion and Writing

89. Draw the graph of a function that has the following properties: domain: all real numbers; range: all real numbers; intercepts: $(0, -3)$ and $(3, 0)$; a local maximum value of -2 is at -1; a local minimum value of -6 is at 2. Compare your graph with those of others. Comment on any differences.

90. Redo Problem 89 with the following additional information: increasing on $(-\infty, -1)$, $(2, \infty)$; decreasing on $(-1, 2)$. Again compare your graph with others and comment on any differences.

91. How many x-intercepts can a function defined on an interval have if it is increasing on that interval? Explain.

92. Suppose that a friend of yours does not understand the idea of increasing and decreasing functions. Provide an explanation, complete with graphs, that clarifies the idea.

93. Can a function be both even and odd? Explain.

94. Using a graphing utility, graph $y = 5$ on the interval $(-3, 3)$. Use MAXIMUM to find the local maximum values on $(-3, 3)$. Comment on the result provided by the calculator.

95. A function f has a positive average rate of change on the interval $[2, 5]$. Is f increasing on $[2, 5]$? Explain.

96. Show that a constant function $f(x) = b$ has an average rate of change of 0. Compute the average rate of change of $y = \sqrt{4 - x^2}$ on the interval $[-2, 2]$. Explain how this can happen.

'Are You Prepared?' Answers

1. $2 < x < 5$ **2.** 1 **3.** symmetric with respect to the y-axis **4.** $y + 2 = 5(x - 3)$ **5.** $(-3, 0), (3, 0), (0, -9)$

1.4 Library of Functions; Piecewise-defined Functions

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Intercepts (Foundations, Section 2, pp. 11–12)
- Graphs of Key Equations (Foundations, Section 2: Example 3, p. 10; Example 8, p. 14 Example 9, p. 15, Example 10, p. 15)

Now Work the 'Are You Prepared?' problems on page 85.

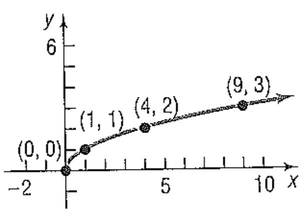
OBJECTIVES 1 Graph the Functions Listed in the Library of Functions (p. 78)
2 Graph Piecewise-defined Functions (p. 83)

1 Graph the Functions Listed in the Library of Functions

First we introduce a few more functions, beginning with the *square root function.*

On page 15, we graphed the equation $y = \sqrt{x}$. Figure 28 shows a graph of the function $f(x) = \sqrt{x}$. Based on the graph, we have the following properties:

Figure 28



Properties of $f(x) = \sqrt{x}$

1. The domain and the range are the set of nonnegative real numbers.
2. The x-intercept of the graph of $f(x) = \sqrt{x}$ is 0. The y-intercept of the graph of $f(x) = \sqrt{x}$ is also 0.
3. The function is neither even nor odd.
4. The function is increasing on the interval $(0, \infty)$.
5. The function has an absolute minimum of 0 at $x = 0$.

EXAMPLE 1 **Graphing the Cube Root Function**

(a) Determine whether $f(x) = \sqrt[3]{x}$ is even, odd, or neither. State whether the graph of f is symmetric with respect to the y-axis or symmetric with respect to the origin.
(b) Determine the intercepts, if any, of the graph of $f(x) = \sqrt[3]{x}$.
(c) Graph $f(x) = \sqrt[3]{x}$.

Solution (a) Because

$$f(-x) = \sqrt[3]{-x} = -\sqrt[3]{x} = -f(x)$$

the function is odd. The graph of f is symmetric with respect to the origin.

(b) The y-intercept is $f(0) = \sqrt[3]{0} = 0$. The x-intercept is found by solving the equation $f(x) = 0$.

$$f(x) = 0$$
$$\sqrt[3]{x} = 0 \quad f(x) = \sqrt[3]{x}$$
$$x = 0 \quad \text{Cube both sides of the equation.}$$

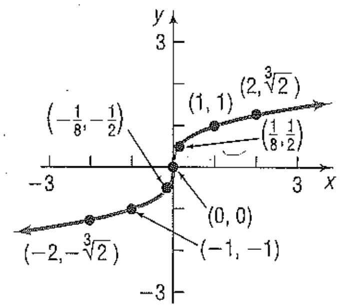
The x-intercept is also 0.

(c) Use the function to form Table 4 and obtain some points on the graph. Because of the symmetry with respect to the origin, we find only points (x, y) for which $x \geq 0$. Figure 29 shows the graph of $f(x) = \sqrt[3]{x}$.

Table 4

x	$y = f(x) = \sqrt[3]{x}$	(x, y)
0	0	$(0, 0)$
$\frac{1}{8}$	$\frac{1}{2}$	$\left(\frac{1}{8}, \frac{1}{2}\right)$
1	1	$(1, 1)$
2	$\sqrt[3]{2} \approx 1.26$	$(2, \sqrt[3]{2})$
8	2	$(8, 2)$

Figure 29



From the results of Example 1 and Figure 29, we have the following properties of the cube root function.

Properties of $f(x) = \sqrt[3]{x}$

1. The domain and the range are the set of all real numbers.
2. The x-intercept of the graph of $f(x) = \sqrt[3]{x}$ is 0. The y-intercept of the graph of $f(x) = \sqrt[3]{x}$ is also 0.
3. The graph is symmetric with respect to the origin. The function is odd.
4. The function is increasing on the interval $(-\infty, \infty)$.
5. The function does not have any local minima or any local maxima.

EXAMPLE 2

Graphing the Absolute Value Function

(a) Determine whether $f(x) = |x|$ is even, odd, or neither. State whether the graph of f is symmetric with respect to the y-axis or symmetric with respect to the origin.

(b) Determine the intercepts, if any, of the graph of $f(x) = |x|$.

(c) Graph $f(x) = |x|$.

Solution

(a) Because

$$f(-x) = |-x| = |x| = f(x)$$

the function is even. The graph of f is symmetric with respect to the y-axis.

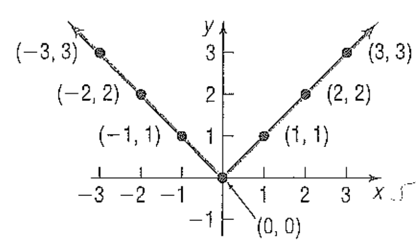
(b) The y-intercept is $f(0) = |0| = 0$. The x-intercept is found by solving the equation $f(x) = 0$ or $|x| = 0$. So the x-intercept is 0.

(c) Use the function to form Table 5 and obtain some points on the graph. Because of the symmetry with respect to the y-axis, we need to find only points (x, y) for which $x \geq 0$. Figure 30 shows the graph of $f(x) = |x|$.

Table 5

x	$y = f(x) = \|x\|$	(x, y)
0	0	(0, 0)
1	1	(1, 1)
2	2	(2, 2)
3	3	(3, 3)

Figure 30



From the results of Example 2 and Figure 30, we have the following properties of the absolute value function.

Properties of $f(x) = |x|$

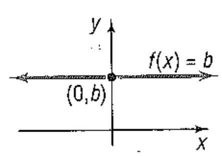
1. The domain is the set of all real numbers. The range of f is $\{y \mid y \geq 0\}$.
2. The x-intercept of the graph of $f(x) = |x|$ is 0. The y-intercept of the graph of $f(x) = |x|$ is also 0.
3. The graph is symmetric with respect to the y-axis. The function is even.
4. The function is decreasing on the interval $(-\infty, 0)$. It is increasing on the interval $(0, \infty)$.
5. The function has an absolute minimum of 0 at $x = 0$.

Seeing the Concept

Graph $y = |x|$ on a square screen and compare what you see with Figure 30. Note that some graphing calculators use abs(x) for absolute value.

Below is a list of the key functions that we have discussed. In going through this list, pay special attention to the properties of each function, particularly to the shape of each graph. Knowing these graphs, along with key points on each graph, will lay the foundation for further graphing techniques.

Figure 31 Constant Function



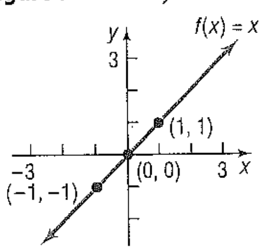
Constant Function

$$f(x) = b \qquad b \text{ is a real number}$$

See Figure 31.

The domain of a **constant function** is the set of all real numbers; its range is the set consisting of a single number b. Its graph is a horizontal line whose y-intercept is b. The constant function is an even function.

Figure 32 Identity Function



Identity Function

$$f(x) = x$$

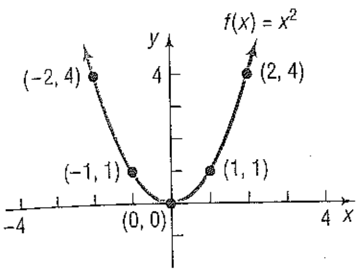
See Figure 32.

The domain and the range of the **identity function** are the set of all real numbers. Its graph is a line whose slope is 1 and whose y-intercept is 0. The line consists of all points for which the x-coordinate equals the y-coordinate. The identity function is an odd function that is increasing over its domain. Note that the graph bisects quadrants I and III.

Square Function

$$f(x) = x^2$$

Figure 33 Square Function



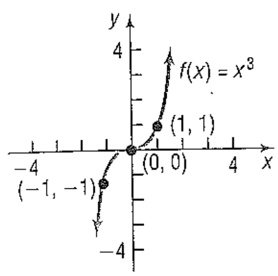
See Figure 33.

The domain of the **square function** is the set of all real numbers; its range is the set of nonnegative real numbers. The graph of this function is a parabola whose vertex is at $(0, 0)$, which is also the only intercept. The square function is an even function that is decreasing on the interval $(-\infty, 0)$ and increasing on the interval $(0, \infty)$.

Cube Function

$$f(x) = x^3$$

Figure 34 Cube Function



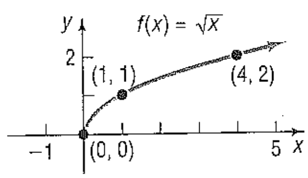
See Figure 34.

The domain and the range of the **cube function** are the set of all real numbers. The intercept of the graph is at $(0, 0)$. The cube function is odd and is increasing on the interval $(-\infty, \infty)$.

Square Root Function

$$f(x) = \sqrt{x}$$

Figure 35 Square Root Function



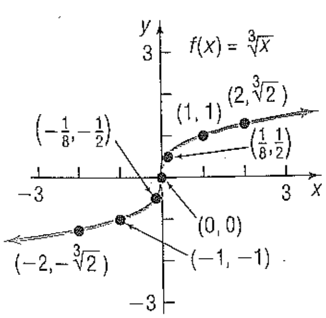
See Figure 35.

The domain and the range of the **square root function** are the set of nonnegative real numbers. The intercept of the graph is at $(0, 0)$. The square root function is neither even nor odd and is increasing on the interval $(0, \infty)$.

Cube Root Function

$$f(x) = \sqrt[3]{x}$$

Figure 36 Cube Root Function



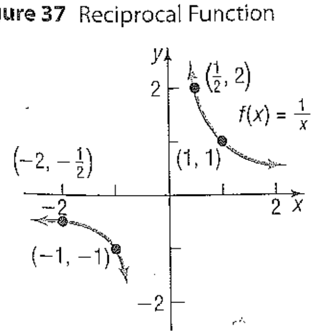
See Figure 36.

The domain and the range of the **cube root function** are the set of all real numbers. The intercept of the graph is at $(0, 0)$. The cube root function is an odd function that is increasing on the interval $(-\infty, \infty)$.

Reciprocal Function

$$f(x) = \frac{1}{x}$$

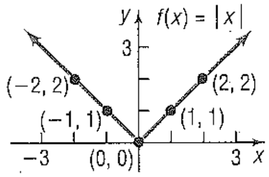
Figure 37 Reciprocal Function



Refer to Example 10, page 15, for a discussion of the equation $y = \frac{1}{x}$. See Figure 37.

The domain and the range of the **reciprocal function** are the set of all nonzero real numbers. The graph has no intercepts. The reciprocal function is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$ and is an odd function.

Figure 38 Absolute Value Function



Absolute Value Function

$$f(x) = |x|$$

See Figure 38.

The domain of the **absolute value function** is the set of all real numbers; it range is the set of nonnegative real numbers. The intercept of the graph is at $(0, 0)$ If $x \geq 0$, then $f(x) = x$, and this part of the graph of f is the line $y = x$; if $x <$ (then $f(x) = -x$, and this part of the graph of f is the line $y = -x$. The absolut value function is an even function; it is decreasing on the interval $(-\infty, 0)$ an increasing on the interval $(0, \infty)$.

The notation $\text{int}(x)$ stands for the largest integer less than or equal to x. Fc example,

$$\text{int}(1) = 1, \quad \text{int}(2.5) = 2, \quad \text{int}\left(\frac{1}{2}\right) = 0, \quad \text{int}\left(-\frac{3}{4}\right) = -1, \quad \text{int}(\pi) = 3$$

This type of correspondence occurs frequently enough in mathematics that we giv it a name.

DEFINITION

Greatest Integer Function

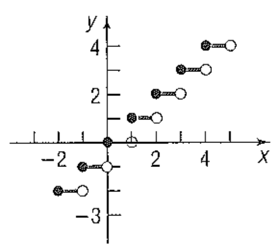
$$f(x) = \text{int}(x)^* = \text{greatest integer less than or equal to } x$$

Table 6

x	$y = f(x) = \text{int}(x)$	(x, y)
-1	-1	$(-1, -1)$
$-\frac{1}{2}$	-1	$\left(-\frac{1}{2}, -1\right)$
$-\frac{1}{4}$	-1	$\left(-\frac{1}{4}, -1\right)$
0	0	$(0, 0)$
$\frac{1}{4}$	0	$\left(\frac{1}{4}, 0\right)$
$\frac{1}{2}$	0	$\left(\frac{1}{2}, 0\right)$
$\frac{3}{4}$	0	$\left(\frac{3}{4}, 0\right)$

We obtain the graph of $f(x) = \text{int}(x)$ by plotting several points. See Table For values of x, $-1 \leq x < 0$, the value of $f(x) = \text{int}(x)$ is -1; for values of $0 \leq x < 1$, the value of f is 0. See Figure 39 for the graph.

Figure 39 Greatest Integer Function

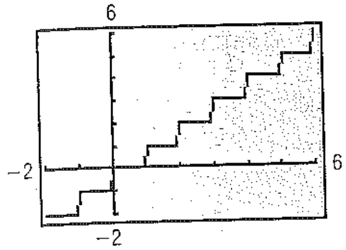


The domain of the **greatest integer function** is the set of all real numbers; i range is the set of integers. The y-intercept of the graph is 0. The x-intercepts l in the interval $[0, 1)$. The greatest integer function is neither even nor odd. It constant on every interval of the form $[k, k + 1)$, for k an integer. In Figure 39, solid dot is used to indicate, for example, that at $x = 1$ the value of f is $f(1) =$ an open circle is used to illustrate that the function does not assume the value of at $x = 1$.

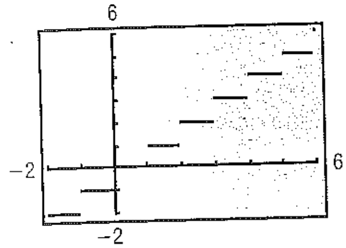
Although a precise definition requires the idea of a limit, discussed in calcul in a rough sense, a function is said to be **continuous** if its graph has no gaps or hol and can be drawn without lifting a pencil from the paper on which the graph drawn. We contrast this with a *discontinuous* function. A function is **discontinuous** its graph has gaps or holes so that its graph cannot be drawn without lifting a pen from the paper.

*Some books use the notation $f(x) = [x]$ instead of $\text{int}(x)$.

Figure 40 $f(x) = \text{int}(x)$



(a) Connected mode



(b) Dot mode

From the graph of the greatest integer function, we can see why it is also called a **step function.** At $x = 0, x = \pm 1, x = \pm 2$, and so on, this function is discontinuous because, at integer values, the graph suddenly "steps" from one value to another without taking on any of the intermediate values. For example, to the immediate left of $x = 3$, the y-coordinates of the points on the graph are 2, and at $x = 3$ and to the immediate right of $x = 3$, the y-coordinates of the points on the graph are 3. Consequently, the graph has gaps in it.

COMMENT When graphing a function using a graphing utility, you can choose either the **connected mode,** in which points plotted on the screen are connected, making the graph appear without any breaks, or the **dot mode,** in which only the points plotted appear. When graphing the greatest integer function with a graphing utility, it may be necessary to be in the dot mode. This is to prevent the utility from "connecting the dots" when $f(x)$ changes from one integer value to the next. See Figure 40. Note that some graphing utilities will display the gaps even when in "connected" mode. ■

The functions discussed so far are basic. Whenever you encounter one of them, you should see a mental picture of its graph. For example, if you encounter the function $f(x) = x^2$, you should see in your mind's eye a picture like Figure 33.

Now Work PROBLEMS 9 THROUGH 16

2 Graph Piecewise-defined Functions

Sometimes a function is defined using different equations on different parts of its domain. For example, the absolute value function $f(x) = |x|$ is defined by two equations: $f(x) = x$ if $x \geq 0$ and $f(x) = -x$ if $x < 0$. For convenience, these equations are generally combined into one expression as

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

When a function is defined by different equations on different parts of its domain, it is called a **piecewise-defined** function.

EXAMPLE 3

Analyzing a Piecewise-defined Function

The function f is defined as

$$f(x) = \begin{cases} -2x + 1 & \text{if } -3 \leq x < 1 \\ 2 & \text{if } x = 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

(a) Find $f(-2), f(1)$, and $f(2)$. (b) Determine the domain of f.
(c) Locate any intercepts. (d) Graph f.
(e) Use the graph to find the range of f. (f) Is f continuous on its domain?

Solution (a) To find $f(-2)$, observe that when $x = -2$, the equation for f is given by $f(x) = -2x + 1$, so

$$f(-2) = -2(-2) + 1 = 5$$

When $x = 1$, the equation for f is $f(x) = 2$. That is,

$$f(1) = 2$$

When $x = 2$, the equation for f is $f(x) = x^2$, so

$$f(2) = 2^2 = 4$$

(b) To find the domain of f, look at its definition. Since f is defined for all x greater than or equal to -3, the domain of f is $\{x | x \geq -3\}$, or the interval $[-3, \infty)$.

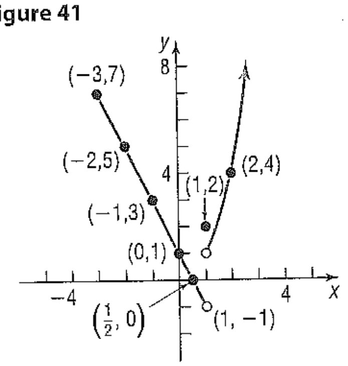
(c) The y-intercept of the graph of the function is $f(0)$. Because the equation for f when $x = 0$ is $f(x) = -2x + 1$, the y-intercept is $f(0) = -2(0) + 1 = 1$.

The x-intercepts of the graph of a function f are the real solutions to the equation $f(x) = 0$. To find the x-intercepts of f, solve $f(x) = 0$ for each "piece" of the function, and then determine whether the values of x, if any, satisfy the condition that defines the piece.

$$\begin{aligned} f(x) &= 0 \\ -2x + 1 &= 0 \quad -3 \le x < 1 \\ -2x &= -1 \\ x &= \frac{1}{2} \end{aligned} \qquad \begin{aligned} f(x) &= 0 \\ 2 &= 0 \quad x = 1 \\ &\text{No solution} \end{aligned} \qquad \begin{aligned} f(x) &= 0 \\ x^2 &= 0 \quad x > 1 \\ x &= 0 \end{aligned}$$

The first potential x-intercept, $x = \frac{1}{2}$, satisfies the condition $-3 \le x < 1$, so $x = \frac{1}{2}$ is an x-intercept. The second potential x-intercept, $x = 0$, does not satisfy the condition $x > 1$, so $x = 0$ is not an x-intercept. The only x-intercept is $\frac{1}{2}$. The intercepts are $(0, 1)$ and $\left(\frac{1}{2}, 0\right)$.

Figure 41



(d) To graph f, graph each "piece." First graph the line $y = -2x + 1$ and keep only the part for which $-3 \le x < 1$. Then plot the point $(1, 2)$ because, when $x = 1$, $f(x) = 2$. Finally, graph the parabola $y = x^2$ and keep only the part for which $x > 1$. See Figure 41.

(e) From the graph, we conclude that the range of f is $\{y \mid y > -1\}$, or the interval $(-1, \infty)$.

(f) The function f is not continuous because there is a "jump" in the graph at $x = 1$.

Now Work PROBLEM 29

EXAMPLE 4

Cost of Electricity

In the spring of 2013, Duke Energy Progress supplied electricity to residences in South Carolina for a monthly customer charge of \$6.50 plus 9.560¢ per kilowatt-hour (kWhr) for the first 800 kWhr supplied in the month and 8.560¢ per kWhr for all usage over 800 kWhr in the month.

(a) What is the charge for using 300 kWhr in a month?
(b) What is the charge for using 1500 kWhr in a month?
(c) If C is the monthly charge for x kWhr, develop a model relating the monthly charge and kilowatt-hours used. That is, express C as a function of x.

Source: Duke Energy Progress, 2013

Solution (a) For 300 kWhr, the charge is \$6.50 plus 9.560¢ = \$0.0956 per kWhr. That is,

$$\text{Charge} = \$6.50 + \$0.0956(300) = \$35.18$$

(b) For 1500 kWhr, the charge is \$6.50 plus 9.560¢ per kWhr for the first 800 kWhr plus 8.560¢ per kWhr for the 700 in excess of 800. That is,

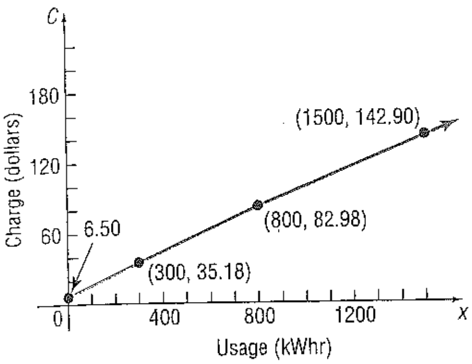
$$\text{Charge} = \$6.50 + \$0.0956(800) + \$0.0856(700) = \$142.90$$

(c) Let x represent the number of kilowatt-hours used. If $0 \le x \le 800$, then the monthly charge C (in dollars) can be found by multiplying x times \$0.0956 and adding the monthly customer charge of \$6.50. Thus, if $0 \le x \le 800$, then $C(x) = 0.0956x + 6.50$.

For $x > 800$, the charge is $0.0956(800) + 6.50 + 0.0856(x - 800)$, since $(x - 800)$ equals the usage in excess of 800 kWhr, which costs \$0.0856 per kWhr. That is, if $x > 800$, then

$$\begin{aligned} C(x) &= 0.0956(800) + 6.50 + 0.0856(x - 800) \\ &= 76.48 + 6.50 + 0.0856x - 68.48 \\ &= 0.0856x + 14.50 \end{aligned}$$

Figure 42



The rule for computing C follows two equations:

$$C(x) = \begin{cases} 0.0956x + 6.50 & \text{if } 0 \le x \le 800 \\ 0.0856x + 14.50 & \text{if } x > 800 \end{cases}$$ The Model

See Figure 42 for the graph.

1.4 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Sketch the graph of $y = \sqrt{x}$. (p. 15)
2. Sketch the graph of $y = \dfrac{1}{x}$. (pp. 15–16)
3. List the intercepts of the equation $y = x^3 - 8$. (p. 12)

Concepts and Vocabulary

4. The function $f(x) = x^2$ is decreasing on the interval ______.
5. When functions are defined by more than one equation, they are called ______ functions.
6. ***True or False*** The cube function is odd and is increasing on the interval $(-\infty, \infty)$.
7. ***True or False*** The cube root function is odd and is decreasing on the interval $(-\infty, \infty)$.
8. ***True or False*** The domain and the range of the reciprocal function are the set of all real numbers.

Skill Building

In Problems 9–16, match each graph to its function.

A. Constant function *B. Identity function* *C. Square function* *D. Cube function*
E. Square root function *F. Reciprocal function* *G. Absolute value function* *H. Cube root function*

9. 

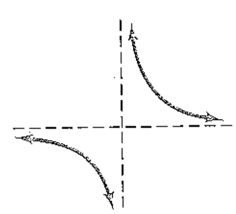
10.

11. 

12. 

13. 

14. 

15. 

16. 

In Problems 17–24, sketch the graph of each function. Be sure to label three points on the graph.

17. $f(x) = x$
18. $f(x) = x^2$
19. $f(x) = x^3$
20. $f(x) = \sqrt{x}$
21. $f(x) = \dfrac{1}{x}$
22. $f(x) = |x|$
23. $f(x) = \sqrt[3]{x}$
24. $f(x) = 3$

25. If $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$

find: (a) $f(-2)$ (b) $f(0)$ (c) $f(2)$

26. If $f(x) = \begin{cases} -3x & \text{if } x < -1 \\ 0 & \text{if } x = -1 \\ 2x^2 + 1 & \text{if } x > -1 \end{cases}$

find: (a) $f(-2)$ (b) $f(-1)$ (c) $f(0)$

27. If $f(x) = \begin{cases} 2x - 4 & \text{if } -1 \le x \le 2 \\ x^3 - 2 & \text{if } 2 < x \le 3 \end{cases}$

find: (a) $f(0)$ (b) $f(1)$ (c) $f(2)$ (d) $f(3)$

28. If $f(x) = \begin{cases} x^3 & \text{if } -2 \le x < 1 \\ 3x + 2 & \text{if } 1 \le x \le 4 \end{cases}$

find: (a) $f(-1)$ (b) $f(0)$ (c) $f(1)$ (d) $f(3)$

In Problems 29–40:

(a) *Find the domain of each function.* (b) *Locate any intercepts.* (c) *Graph each function.*

(d) *Based on the graph, find the range.* (e) *Is f continuous on its domain?*

29. $f(x) = \begin{cases} 2x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

30. $f(x) = \begin{cases} 3x & \text{if } x \neq 0 \\ 4 & \text{if } x = 0 \end{cases}$

31. $f(x) = \begin{cases} -2x + 3 & \text{if } x < 1 \\ 3x - 2 & \text{if } x \geq 1 \end{cases}$

32. $f(x) = \begin{cases} x + 3 & \text{if } x < -2 \\ -2x - 3 & \text{if } x \geq -2 \end{cases}$

33. $f(x) = \begin{cases} x + 3 & \text{if } -2 \leq x < 1 \\ 5 & \text{if } x = 1 \\ -x + 2 & \text{if } x > 1 \end{cases}$

34. $f(x) = \begin{cases} 2x + 5 & \text{if } -3 \leq x < 0 \\ -3 & \text{if } x = 0 \\ -5x & \text{if } x > 0 \end{cases}$

35. $f(x) = \begin{cases} 1 + x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$

36. $f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x < 0 \\ \sqrt[3]{x} & \text{if } x \geq 0 \end{cases}$

37. $f(x) = \begin{cases} |x| & \text{if } -2 \leq x < 0 \\ x^3 & \text{if } x \geq 0 \end{cases}$

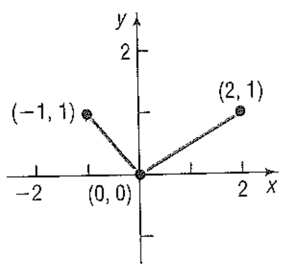
38. $f(x) = \begin{cases} 2 - x & \text{if } -3 \leq x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$

39. $f(x) = 2\,\text{int}(x)$

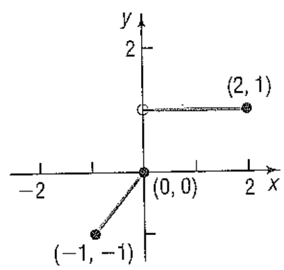
40. $f(x) = \text{int}(2x)$

In Problems 41–44, the graph of a piecewise-defined function is given. Write a definition for each function.

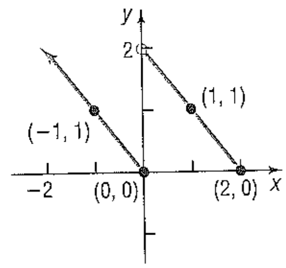
41.



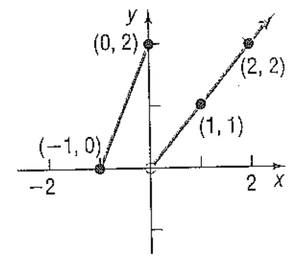
42.



43.



44.



45. If $f(x) = \text{int}(2x)$, find
(a) $f(1.2)$ (b) $f(1.6)$ (c) $f(-1.8)$

46. If $f(x) = \text{int}\left(\dfrac{x}{2}\right)$, find
(a) $f(1.2)$ (b) $f(1.6)$ (c) $f(-1.8)$

Applications and Extensions

47. **Cell Phone Service** Sprint PCS offers a monthly cellular phone plan for \$39.99. It includes 450 anytime minutes and charges \$0.45 per minute for additional minutes. The following function is used to compute the monthly cost for a subscriber:

$$C(x) = \begin{cases} 39.99 & \text{if } 0 \leq x \leq 450 \\ 0.45x - 162.51 & \text{if } x > 450 \end{cases}$$

where x is the number of anytime minutes used. Compute the monthly cost of the cellular phone for use of the following numbers of anytime minutes:
(a) 200 (b) 465 (c) 451

Source: Sprint PCS

48. **Parking at O'Hare International Airport** The short-term (no more than 24 hours) parking fee F (in dollars) for parking x hours on a weekday at O'Hare International Airport's main parking garage can be modeled by the function

$$F(x) = \begin{cases} 2 & \text{if } 0 < x \leq 1 \\ 4 & \text{if } 1 < x \leq 3 \\ 10 & \text{if } 3 < x \leq 4 \\ 5\,\text{int}(x + 1) + 2 & \text{if } 4 < x < 9 \\ 51 & \text{if } 9 \leq x \leq 24 \end{cases}$$

Determine the fee for parking in the short-term parking garage for
(a) 2 hours (b) 7 hours (c) 15 hours
(d) 8 hours and 24 minutes

Source: O'Hare International Airport

49. **Cost of Natural Gas** In March 2013, Peoples Energy had the following rate schedule for natural gas usage in single-family residences:

Monthly service charge	\$22.25
Per therm service charge	
First 50 therms	\$0.25963/therm
Over 50 therms	\$0.11806/therm
Gas charge	\$0.3922/therm

(a) What is the charge for using 50 therms in a month?
(b) What is the charge for using 500 therms in a month?
(c) Develop a function that models the monthly charge for x therms of gas.
(d) Graph the function found in part (c).

Source: Peoples Energy, Chicago, Illinois, 2013

50. **Cost of Natural Gas** In February 2013, Laclede Gas had the following rate schedule for natural gas usage in single-family residences:

Monthly customer charge	\$19.50
Distribution charge	
First 30 therms	\$0.65403/therm
Over 30 therms	\$0.04235/therm
Gas supply charge	\$0.53668/therm

(a) What is the charge for using 20 therms in a month?
(b) What is the charge for using 150 therms in a month?
(c) Develop a model that gives the monthly charge C for x therms of gas.
(d) Graph the function found in part (c).

Source: Laclede Gas, 2013

51. **Federal Income Tax** Refer to the 2013 Tax Rate Schedules. If x equals taxable income and y equals the tax due, construct a function $y = f(x)$ for Schedule X.

52. **Federal Income Tax** Refer to the 2013 tax rate schedules. If x equals taxable income and y equals the tax due, construct a function $y = f(x)$ for Schedule Y-1.

2013 Tax Rate Schedules

Schedule X-Single						Schedule Y-I - Married Filing Jointly or Qualified Widow(er)					
If Taxable Income Is Over	But Not Over	The Tax Is This Amount		Plus This %	Of the Excess Over	If Taxable Income Is Over	But Not Over	The Tax Is This Amount		Plus This %	Of the Excess Over
\$0	\$8,925	\$0	+	10%	\$0	\$0	\$17,850	\$0	+	10%	\$0
8,925	36,250	892.50	+	15%	8,925	17,850	72,500	1,785	+	15%	17,850
36,250	87,850	4,991.25	+	25%	36,250	72,500	146,400	9,982.50	+	25%	72,500
87,850	183,250	17,891.25	+	28%	87,850	146,400	223,050	28,457.50	+	28%	146,400
183,250	398,350	44,603.25	+	33%	183,250	223,050	398,350	49,919.50	+	33%	223,050
398,350	400,000	115,586.25	+	35%	398,350	398,350	450,000	107,768.50	+	35%	398,350
400,000	–	116,163.75	+	39.6%	400,000	450,000	–	125,846	+	39.6%	450,000

Source: Internal Revenue Service

53. **Cost of Transporting Goods** A trucking company transports goods between Chicago and New York, a distance of 960 miles. The company's policy is to charge, for each pound, \$0.50 per mile for the first 100 miles, \$0.40 per mile for the next 300 miles, \$0.25 per mile for the next 400 miles, and no charge for the remaining 160 miles.
(a) Graph the relationship between the cost of transportation in dollars and mileage over the entire 960-mile route.
(b) Find the cost as a function of mileage for hauls between 100 and 400 miles from Chicago.
(c) Find the cost as a function of mileage for hauls between 400 and 800 miles from Chicago.

54. **Car Rental Costs** An economy car rented in Florida from Enterprise® on a weekly basis costs \$185 per week. Extra days cost \$37 per day until the day rate exceeds the weekly rate, in which case the weekly rate applies. Also, any part of a day used counts as a full day. Find the cost C of renting an economy car as a function of the number x of days used, where $7 \le x \le 14$. Graph this function.

Source: enterprise.com

55. **Mortgage Fees** Fannie Mae charges a loan-level price adjustment (LLPA) on all mortgages, which represents a fee homebuyers seeking a loan must pay. The rate paid depends on the credit score of the borrower, the amount borrowed, and the loan-to-value (LTV) ratio. The LTV ratio is the ratio of amount borrowed to appraised value of the home. For example, a homebuyer who wishes to borrow \$250,000 with a credit score of 730 and an LTV ratio of 80% will pay 0.5% (0.005) of \$250,000, or \$1250. The table shows the LLPA for various credit scores and an LTV ratio of 80%.

Credit Score	Loan-Level Price Adjustment Rate
≤ 659	3.00%
660–679	2.50%
680–699	1.75%
700–719	1%
720–739	0.5%
≥ 740	0.25%

Source: Fannie Mae.

(a) Construct a function $C = C(s)$, where C is the loan-level price adjustment (LLPA) and s is the credit score of an individual who wishes to borrow \$300,000 with an 80% LTV ratio.
(b) What is the LLPA on a \$300,000 loan with an 80% LTV ratio for a borrower whose credit score is 725?
(c) What is the LLPA on a \$300,000 loan with an 80% LTV ratio for a borrower whose credit score is 670?

56. **Minimum Payments for Credit Cards** Holders of credit cards issued by banks, department stores, oil companies, and so on receive bills each month that state minimum amounts that must be paid by a certain due date. The minimum due depends on the total amount owed. One such credit card

company uses the following rules: For a bill of less than \$10, the entire amount is due. For a bill of at least \$10 but less than \$500, the minimum due is \$10. A minimum of \$30 is due on a bill of at least \$500 but less than \$1000, a minimum of \$50 is due on a bill of at least \$1000 but less than \$1500, and a minimum of \$70 is due on bills of \$1500 or more. Find the function f that describes the minimum payment due on a bill of x dollars. Graph f.

57. **Wind Chill** The wind chill factor represents the equivalent air temperature at a standard wind speed that would produce the same heat loss as the given temperature and wind speed. One formula for computing the equivalent temperature is

$$W = \begin{cases} t & 0 \le v < 1.79 \\ 33 - \dfrac{(10.45 + 10\sqrt{v} - v)(33 - t)}{22.04} & 1.79 \le v \le 20 \\ 33 - 1.5958(33 - t) & v > 20 \end{cases}$$

where v represents the wind speed (in meters per second) and t represents the air temperature (°C). Compute the wind chill for the following:

(a) An air temperature of 10°C and a wind speed of 1 meter per second (m/sec)
(b) An air temperature of 10°C and a wind speed of 5 m/sec
(c) An air temperature of 10°C and a wind speed of 15 m/sec
(d) An air temperature of 10°C and a wind speed of 25 m/sec
(e) Explain the physical meaning of the equation corresponding to $0 \le v < 1.79$.
(f) Explain the physical meaning of the equation corresponding to $v > 20$.

58. **Wind Chill** Redo Problem 57(a) – (d) for an air temperature of −10°C.

59. **First-class Mail** In 2013 the U.S. Postal Service charged \$0.92 postage for first-class mail retail flats (such as an 8.5" by 11" envelope) weighing up to 1 ounce, plus \$0.20 for each additional ounce up to 13 ounces. First-class rates do not apply to flats weighing more than 13 ounces. Develop a model that relates C, the first-class postage charged, for a flat weighing x ounces. Graph the function.

Source: United States Postal Service

Discussion and Writing

In Problems 60–67, use a graphing utility.

60. **Exploration** Graph $y = x^2$. Then on the same screen graph $y = x^2 + 2$, followed by $y = x^2 + 4$, followed by $y = x^2 - 2$. What pattern do you observe? Can you predict the graph of $y = x^2 - 4$? Of $y = x^2 + 5$?

61. **Exploration** Graph $y = x^2$. Then on the same screen graph $y = (x - 2)^2$, followed by $y = (x - 4)^2$, followed by $y = (x + 2)^2$. What pattern do you observe? Can you predict the graph of $y = (x + 4)^2$? Of $y = (x - 5)^2$?

62. **Exploration** Graph $y = |x|$. Then on the same screen graph $y = 2|x|$, followed by $y = 4|x|$, followed by $y = \frac{1}{2}|x|$. What pattern do you observe? Can you predict the graph of $y = \frac{1}{4}|x|$? Of $y = 5|x|$?

63. **Exploration** Graph $y = x^2$. Then on the same screen graph $y = -x^2$. What pattern do you observe? Now try $y = |x|$ and $y = -|x|$. What do you conclude?

64. **Exploration** Graph $y = \sqrt{x}$. Then on the same screen graph $y = \sqrt{-x}$. What pattern do you observe? Now try $y = 2x + 1$ and $y = 2(-x) + 1$. What do you conclude?

65. **Exploration** Graph $y = x^3$. Then on the same screen graph $y = (x - 1)^3 + 2$. Could you have predicted the result?

66. **Exploration** Graph $y = x^2$, $y = x^4$, and $y = x^6$ on the same screen. What do you notice is the same about each graph? What do you notice is different?

67. **Exploration** Graph $y = x^3$, $y = x^5$, and $y = x^7$ on the same screen. What do you notice is the same about each graph? What do you notice is different?

68. Consider the equation

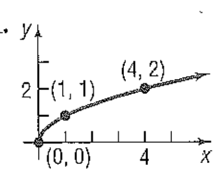
$$y = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Is this a function? What is its domain? What is its range? What is its y-intercept, if any? What are its x-intercepts, if any? Is it even, odd, or neither? How would you describe its graph?

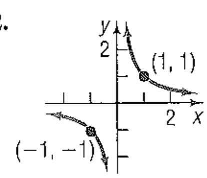
69. Define some functions that pass through $(0, 0)$ and $(1, 1)$ and are increasing for $x \ge 0$. Begin your list with $y = \sqrt{x}$, $y = x$, and $y = x^2$. Can you propose a general result about such functions?

'Are You Prepared?' Answers

1.



2.



3. $(0, -8)$, $(2, 0)$

1.5 Graphing Techniques: Transformations

OBJECTIVES
1 Graph Functions Using Vertical and Horizontal Shifts (p. 89)
2 Graph Functions Using Compressions and Stretches (p. 92)
3 Graph Functions Using Reflections about the x-Axis and the y-Axis (p. 94)

At this stage, if you were asked to graph any of the functions defined by $y = x$, $y = x^2$, $y = x^3$, $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = \frac{1}{x}$, or $y = |x|$, your response should be "Yes, I recognize these functions and know the general shapes of their graphs." (If this is not your answer, review the previous section, Figures 32 through 38.)

Sometimes we are asked to graph a function that is "almost" like one that we already know how to graph. In this section, we develop techniques for graphing such functions. Collectively, these techniques are referred to as **transformations.**

1 Graph Functions Using Vertical and Horizontal Shifts

EXAMPLE 1 Vertical Shift Up

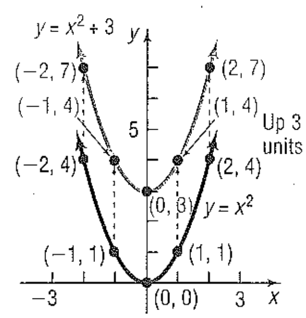
Use the graph of $f(x) = x^2$ to obtain the graph of $g(x) = x^2 + 3$. Find the domain and range of g.

Solution Begin by obtaining some points on the graphs of f and g. For example, when $x = 0$, then $y = f(0) = 0$ and $y = g(0) = 3$. When $x = 1$, then $y = f(1) = 1$ and $y = g(1) = 4$. Table 7 lists these and a few other points on each graph. Notice that each y-coordinate of a point on the graph of g is 3 units larger than the y-coordinate of the corresponding point on the graph of f. We conclude that the graph of g is identical to that of f, except that it is shifted vertically up 3 units. See Figure 43.

Table 7

x	$y = f(x) = x^2$	$y = g(x) = x^2 + 3$
−2	4	7
−1	1	4
0	0	3
1	1	4
2	4	7

Figure 43



The domain of g is all real numbers, or $(-\infty, \infty)$. The range of g is $[3, \infty)$.

EXAMPLE 2 Vertical Shift Down

Use the graph of $f(x) = x^2$ to obtain the graph of $g(x) = x^2 - 4$. Find the domain and range of g.

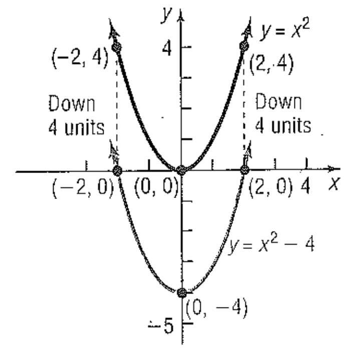
Solution Table 8 on the next page lists some points on the graphs of f and g. Notice that each y-coordinate of g is 4 units less than the corresponding y-coordinate of f.

To obtain the graph of g from the graph of f, subtract 4 from each y-coordinate on the graph of f. The graph of g is identical to that of f, except that it is shifted down 4 units. See Figure 44 on the next page.

Table 8

x	$y = f(x) = x^2$	$y = g(x) = x^2 - 4$
−2	4	0
−1	1	−3
0	0	−4
1	1	−3
2	4	0

Figure 44



The domain of g is all real numbers, or $(-\infty, \infty)$. The range of g is $[-4, \infty)$.

Note that a vertical shift affects only the range of a function, not the domain. For example, the range of $f(x) = x^2$ is $[0, \infty)$. In Example 1 the range of g is $[3, \infty)$, whereas in Example 2 the range of g is $[-4, \infty)$. The domain for all three functions is all real numbers.

We are led to the following conclusions:

If a positive real number k is added to the output of a function $y = f(x)$, the graph of the new function $y = f(x) + k$ is the graph of f **shifted vertically up** k units.

If a positive real number k is subtracted from the output of a function $y = f(x)$, the graph of the new function $y = f(x) - k$ is the graph of f **shifted vertically down** k units.

Now Work PROBLEM 39

EXAMPLE 3 Horizontal Shift to the Right

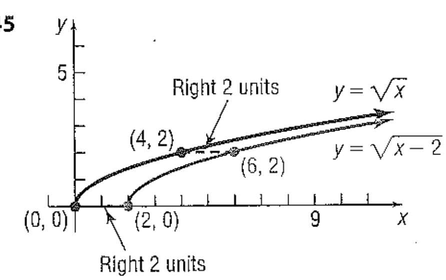
Use the graph of $f(x) = \sqrt{x}$ to obtain the graph of $g(x) = \sqrt{x-2}$. Find the domain and range of g.

Solution The function $g(x) = \sqrt{x-2}$ is basically a square root function. Table 9 lists some points on the graphs of f and g. Note that when $f(x) = 0$ then $x = 0$, and when $g(x) = 0$, then $x = 2$. Also, when $f(x) = 2$, then $x = 4$, and when $g(x) = 2$, then $x = 6$. Notice that the x-coordinates on the graph of g are 2 units larger than the corresponding x-coordinates on the graph of f for any given y-coordinate. We conclude that the graph of g is identical to that of f, except that it is shifted horizontally 2 units to the right. See Figure 45.

Table 9

x	$y = f(x) = \sqrt{x}$	x	$y = g(x) = \sqrt{x-2}$
0	0	2	0
1	1	3	1
4	2	6	2
9	3	11	3

Figure 45



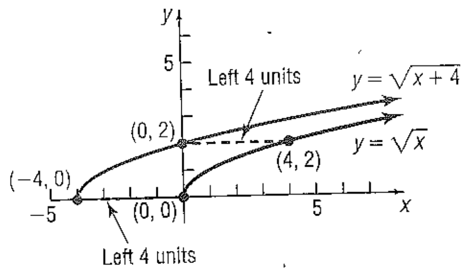
The domain of g is $[2, \infty)$ and the range is $[0, \infty)$.

EXAMPLE 4 Horizontal Shift to the Left

Use the graph of $f(x) = \sqrt{x}$ to obtain the graph of $g(x) = \sqrt{x+4}$. Find the domain and range of g.

Solution Again, the function $g(x) = \sqrt{x+4}$ is basically a square root function. Its graph is the same as that of f, except that it is shifted horizontally 4 units to the left. See Figure 46.

Figure 46



The domain of g is $[-4, \infty)$ and the range is $[0, \infty)$.

Now Work PROBLEM 43

Note that a horizontal shift affects only the domain of a function, not the range. For example, the domain of $f(x) = \sqrt{x}$ is $[0, \infty)$. In Example 3 the domain of g is $[2, \infty)$, whereas in Example 4 the domain of g is $[-4, \infty)$. The range for all three functions is $[0, \infty)$.

We are led to the following conclusion.

> **In Words**
> For $y = f(x - h)$, add h to each x-coordinate on the graph of $y = f(x)$. For $y = f(x + h)$, subtract h from each x-coordinate on the graph of $y = f(x)$.

If the argument x of a function f is replaced by $x - h$, $h > 0$, the graph of the new function $y = f(x - h)$ is the graph of f **shifted horizontally right** h units.

If the argument x of a function f is replaced by $x + h$, $h > 0$, the graph of the new function $y = f(x + h)$ is the graph of f **shifted horizontally left** h units.

Observe the distinction between vertical and horizontal shifts. The graph of $f(x) = x^3 + 2$ is obtained by shifting the graph of $y = x^3$ *up* 2 units, because we evaluate the cube function first and then add 2. The graph of $g(x) = (x + 2)^3$ is obtained by shifting the graph of $y = x^3$ *left* 2 units, because we add 2 to x before we evaluate the cube function.

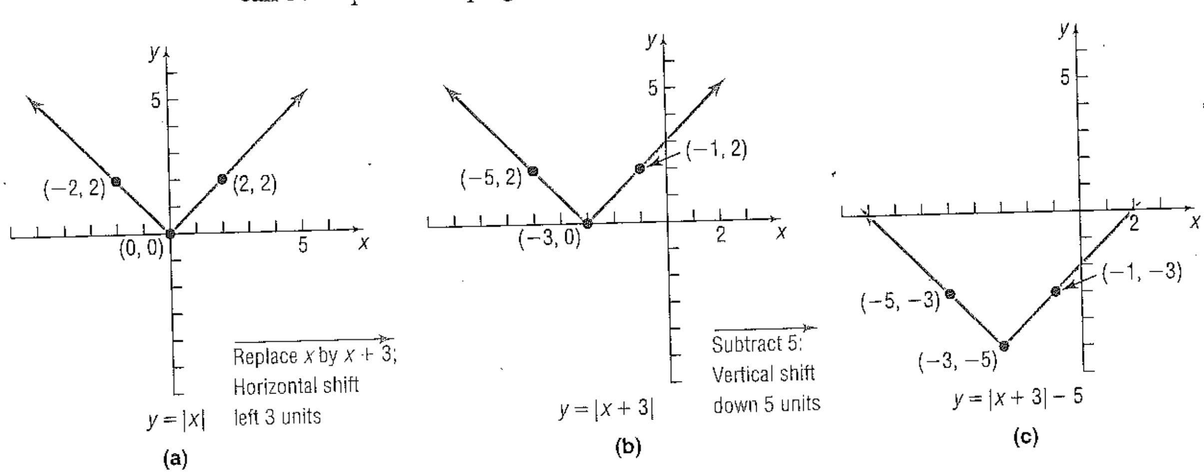
Vertical and horizontal shifts are sometimes combined.

EXAMPLE 5 **Combining Vertical and Horizontal Shifts**

Graph the function $f(x) = |x + 3| - 5$. Find the domain and range of f.

Solution We graph f in steps. First, note that the rule for f is basically an absolute value function, so begin with the graph of $y = |x|$ as shown in Figure 47(a). Next, to get the graph of $y = |x + 3|$, shift the graph of $y = |x|$ horizontally 3 units to the left. See Figure 47(b). Finally, to get the graph of $y = |x + 3| - 5$, shift the graph of $y = |x + 3|$ vertically down 5 units. See Figure 47(c). Note the points plotted on each graph. Using key points can be helpful in keeping track of the transformation that has taken place.

Figure 47



The domain of f is all real numbers, or $(-\infty, \infty)$. The range of f is $[-5, \infty)$.

Check: Graph $Y_1 = f(x) = |x + 3| - 5$ and compare the graph to Figure 47(c).

In Example 5, if the vertical shift had been done first, followed by the horizontal shift, the final graph would have been the same. Try it for yourself.

Now Work PROBLEMS 45 AND 75

2 Graph Functions Using Compressions and Stretches

EXAMPLE 6 Vertical Stretch

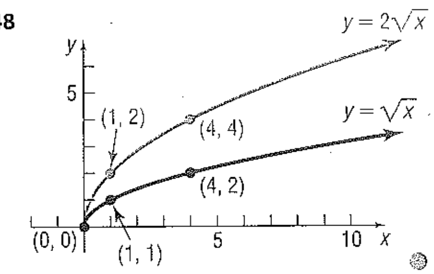
Use the graph of $f(x) = \sqrt{x}$ to obtain the graph of $g(x) = 2\sqrt{x}$.

Solution To see the relationship between the graphs of f and g, we form Table 10, listing points on each graph. For each x, the y-coordinate of a point on the graph of g is 2 times as large as the corresponding y-coordinate on the graph of f. The graph of $f(x) = \sqrt{x}$ is vertically stretched by a factor of 2 to obtain the graph of $g(x) = 2\sqrt{x}$ [for example, $(1, 1)$ is on the graph of f, but $(1, 2)$ is on the graph of g]. See Figure 48.

Table 10

x	$y = f(x) = \sqrt{x}$	$y = g(x) = 2\sqrt{x}$
0	0	0
1	1	2
4	2	4
9	3	6

Figure 48



EXAMPLE 7 Vertical Compression

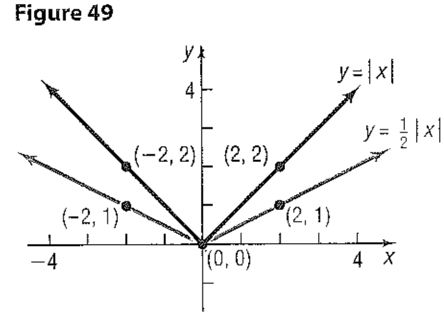
Use the graph of $f(x) = |x|$ to obtain the graph of $g(x) = \frac{1}{2}|x|$.

Solution For each x, the y-coordinate of a point on the graph of g is $\frac{1}{2}$ as large as the corresponding y-coordinate on the graph of f. The graph of $f(x) = |x|$ is vertically compressed by a factor of $\frac{1}{2}$ to obtain the graph of $g(x) = \frac{1}{2}|x|$. For example, $(2, 2)$ is on the graph of f, but $(2, 1)$ is on the graph of g. See Table 11 and Figure 49.

Table 11

x	$y = f(x) = \lvert x\rvert$	$y = g(x) = \frac{1}{2}\lvert x\rvert$
−2	2	1
−1	1	$\frac{1}{2}$
0	0	0
1	1	$\frac{1}{2}$
2	2	1

Figure 49



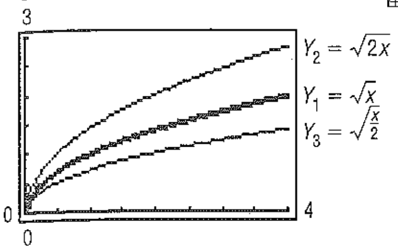
> **In Words**
> For $y = af(x)$, the factor a is "outside" the function, so it affects the y-coordinates. Multiply each y-coordinate on the graph of $y = f(x)$ by a.

When the right side of a function $y = f(x)$ is multiplied by a positive number a, the graph of the new function $y = af(x)$ is obtained by multiplying each y-coordinate on the graph of $y = f(x)$ by a. The new graph is a **vertically compressed** (if $0 < a < 1$) or a **vertically stretched** (if $a > 1$) version of the graph of $y = f(x)$.

Now Work PROBLEM 47

What happens if the argument x of a function $y = f(x)$ is multiplied by a positive number a, creating a new function $y = f(ax)$? To find the answer, look at the following Exploration.

Figure 50



Exploration On the same screen, graph each of the following functions:

$$Y_1 = f(x) = \sqrt{x} \qquad Y_2 = f(2x) = \sqrt{2x} \qquad Y_3 = f\left(\frac{1}{2}x\right) = \sqrt{\frac{1}{2}x} = \sqrt{\frac{x}{2}}$$

Create a table of values to explore the relation between the x- and y-coordinates of each function.

Result You should have obtained the graphs in Figure 50. Look at Table 12(a). Notice that (1, 1), (4, 2), and (9, 3) are points on the graph of $Y_1 = \sqrt{x}$. Also, (0.5, 1), (2, 2), and (4.5, 3) are points on the graph of $Y_2 = \sqrt{2x}$. For a given y-coordinate, the x-coordinate on the graph of Y_2 is $\frac{1}{2}$ of the x-coordinate on Y_1.

Table 12

X	Y1	Y2
0	0	0
.5	.70711	1
1	1	1.4142
2	1.4142	2
4	2	2.8284
4.5	2.1213	3
9	3	4.2426

Y2=√(2X)

X	Y1	Y3
0	0	0
1	1	.70711
2	1.4142	1
4	2	1.4142
8	2.8284	2
9	3	2.1213
18	4.2426	3

Y3=√(X/2)

We conclude that the graph of $Y_2 = \sqrt{2x}$ is obtained by multiplying the x-coordinate of each point on the graph of $Y_1 = \sqrt{x}$ by $\frac{1}{2}$. The graph of $Y_2 = \sqrt{2x}$ is the graph of $Y_1 = \sqrt{x}$ *compressed* horizontally.

Look at Table 12(b). Notice that (1, 1), (4, 2), and (9, 3) are points on the graph of $Y_1 = \sqrt{x}$. Also notice that (2, 1), (8, 2), and (18, 3) are points on the graph of $Y_3 = \sqrt{\frac{x}{2}}$. For a given y-coordinate, the x-coordinate on the graph of Y_3 is 2 times the x-coordinate on Y_1. We conclude that the graph of $Y_3 = \sqrt{\frac{x}{2}}$ is obtained by multiplying the x-coordinate of each point on the graph of $Y_1 = \sqrt{x}$ by 2. The graph of $Y_3 = \sqrt{\frac{x}{2}}$ is the graph of $Y_1 = \sqrt{x}$ *stretched* horizontally.

Based on the Exploration, we have the following result:

> **In Words**
> For $y = f(ax)$, the factor a is "inside" the function, so it affects the x-coordinates. Multiply each x-coordinate on the graph of $y = f(x)$ by $\frac{1}{a}$.

If the argument x of a function $y = f(x)$ is multiplied by a positive number a, then the graph of the new function $y = f(ax)$ is obtained by multiplying each x-coordinate of $y = f(x)$ by $\frac{1}{a}$. A **horizontal compression** results if $a > 1$, and a **horizontal stretch** results if $0 < a < 1$.

Let's look at an example.

EXAMPLE 8

Graphing Using Stretches and Compressions

The graph of $y = f(x)$ is given in Figure 51. Use this graph to find the graphs of

(a) $y = 2f(x)$ (b) $y = f(3x)$

Solution

(a) The graph of $y = 2f(x)$ is obtained by multiplying each y-coordinate of $y = f(x)$ by 2. See Figure 52.

(b) The graph of $y = f(3x)$ is obtained from the graph of $y = f(x)$ by multiplying each x-coordinate of $y = f(x)$ by $\frac{1}{3}$. See Figure 53.

Figure 51

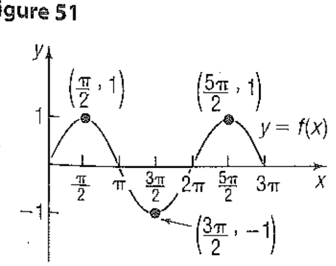
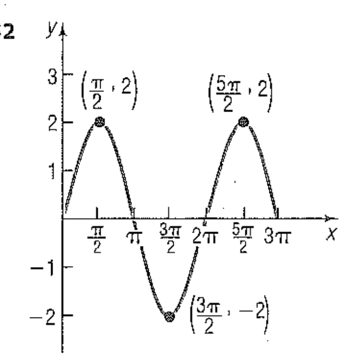
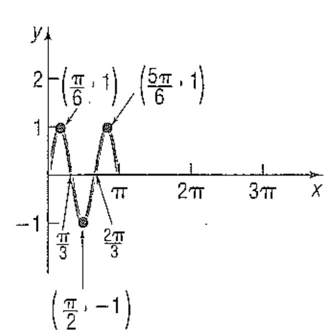


Figure 52



$y = 2f(x)$

Figure 53



$y = f(3x)$

Now Work PROBLEMS 69(e) AND (g)

3 Graph Functions Using Reflections about the x-Axis and the y-Axis

EXAMPLE 9 **Reflection about the x-Axis**

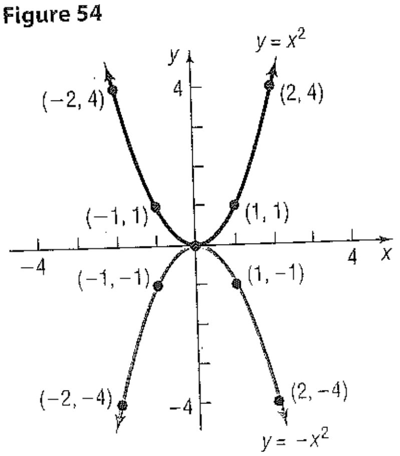
Graph the function $f(x) = -x^2$. Find the domain and range of f.

Solution Begin with the graph of $y = x^2$, as shown in black in Figure 54. For each point (x, y) on the graph of $y = x^2$, the point $(x, -y)$ is on the graph of $y = -x^2$, as indicated in Table 13. Draw the graph of $y = -x^2$ by reflecting the graph of $y = x^2$ about the x-axis. See Figure 54.

Table 13

x	$y = x^2$	$y = -x^2$
−2	4	−4
−1	1	−1
0	0	0
1	1	−1
2	4	−4

Figure 54



The domain of f is all real numbers, or $(-\infty, \infty)$. The range of f is $(-\infty, 0]$.

When the right side of the function $y = f(x)$ is multiplied by -1, the graph of the new function $y = -f(x)$ is the **reflection about the x-axis** of the graph of the function $y = f(x)$.

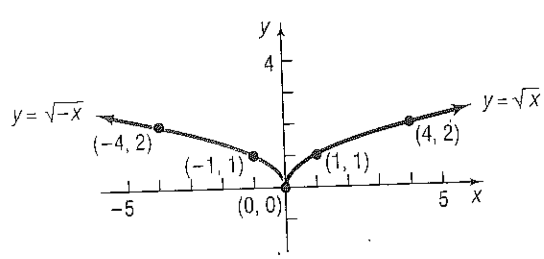
Now Work PROBLEM 51

EXAMPLE 10 **Reflection about the y-Axis**

Graph the function $f(x) = \sqrt{-x}$. Find the domain and range of f.

Solution To get the graph of $f(x) = \sqrt{-x}$, begin with the graph of $y = \sqrt{x}$, as shown in Figure 55. For each point (x, y) on the graph of $y = \sqrt{x}$, the point $(-x, y)$ is on the graph of $y = \sqrt{-x}$. Obtain the graph of $y = \sqrt{-x}$ by reflecting the graph of $y = \sqrt{x}$ about the y-axis. See Figure 55.

Figure 55



The domain of f is $(-\infty, 0]$. The range of f is $[0, \infty)$.

In Words
For $y = -f(x)$, multiply each y-coordinate on the graph of $y = f(x)$ by -1.
For $y = f(-x)$, multiply each x-coordinate by -1.

When the graph of the function $y = f(x)$ is known, the graph of the new function $y = f(-x)$ is the **reflection about the y-axis** of the graph of the function $y = f(x)$.

SUMMARY OF GRAPHING TECHNIQUES

To Graph:	Draw the Graph of f and:	Functional Change to $f(x)$
Vertical shifts		
$y = f(x) + k, \quad k > 0$	Raise the graph of f by k units.	Add k to $f(x)$.
$y = f(x) - k, \quad k > 0$	Lower the graph of f by k units.	Subtract k from $f(x)$.
Horizontal shifts		
$y = f(x + h), \quad h > 0$	Shift the graph of f to the left h units.	Replace x by $x + h$.
$y = f(x - h), \quad h > 0$	Shift the graph of f to the right h units.	Replace x by $x - h$.
Compressing or stretching		
$y = af(x), \quad a > 0$	Multiply each y-coordinate of $y = f(x)$ by a. Stretch the graph of f vertically if $a > 1$. Compress the graph of f vertically if $0 < a < 1$.	Multiply $f(x)$ by a.
$y = f(ax), \quad a > 0$	Multiply each x-coordinate of $y = f(x)$ by $\frac{1}{a}$. Stretch the graph of f horizontally if $0 < a < 1$. Compress the graph of f horizontally if $a > 1$.	Replace x by ax.
Reflection about the x-axis		
$y = -f(x)$	Reflect the graph of f about the x-axis.	Multiply $f(x)$ by -1.
Reflection about the y-axis		
$y = f(-x)$	Reflect the graph of f about the y-axis.	Replace x by $-x$.

EXAMPLE 11 **Determining the Function Obtained from a Series of Transformations**

Find the function that is finally graphed after the following three transformations are applied to the graph of $y = |x|$.

1. Shift left 2 units
2. Shift up 3 units
3. Reflect about the y-axis

Solution

1. Shift left 2 units:	Replace x by $x + 2$.	$y = \|x + 2\|$
2. Shift up 3 units:	Add 3.	$y = \|x + 2\| + 3$
3. Reflect about the y-axis:	Replace x by $-x$.	$y = \|-x + 2\| + 3$

Now Work PROBLEM 27

The examples that follow combine some of the procedures outlined in this section to get the required graph.

EXAMPLE 12 **Combining Graphing Procedures**

Graph the function $f(x) = \dfrac{3}{x - 2} + 1$. Find the domain and range of f.

Solution It is helpful to write f as $f(x) = 3\left(\frac{1}{x-2}\right) + 1$. Now use the following steps to obtain the graph of f:

Step 1: $y = \frac{1}{x}$ Reciprocal function

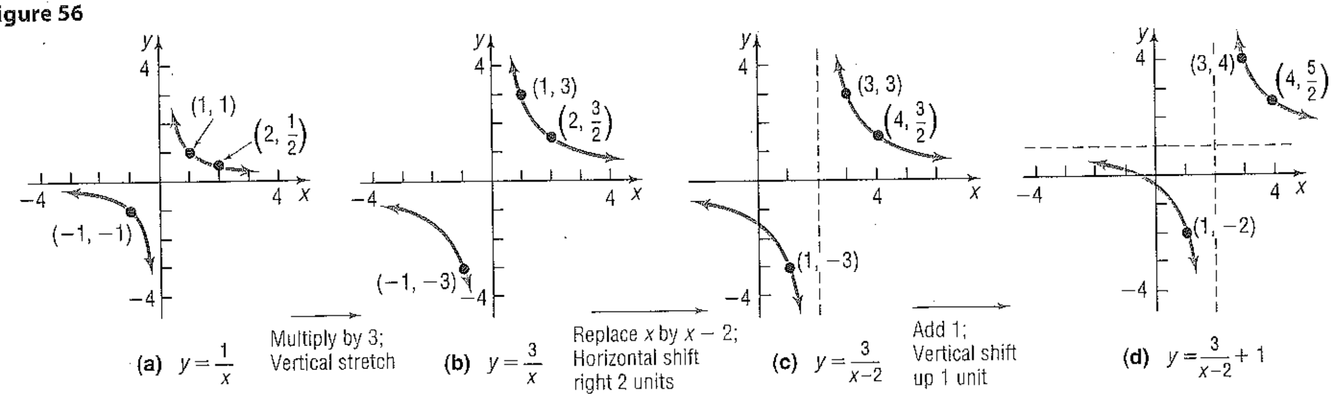
Step 2: $y = 3 \cdot \left(\frac{1}{x}\right) = \frac{3}{x}$ Multiply by 3; vertical stretch of the graph of $y = \frac{1}{x}$ by a factor of 3.

Step 3: $y = \frac{3}{x-2}$ Replace x by x − 2; horizontal shift to the right 2 units.

Step 4: $y = \frac{3}{x-2} + 1$ Add 1; vertical shift up 1 unit.

See Figure 56.

Figure 56



The domain of $y = \frac{1}{x}$ is $\{x|x \neq 0\}$ and its range is $\{y|y \neq 0\}$. Because we shifted right 2 units and up 1 unit to obtain f, the domain of f is $\{x|x \neq 2\}$ and its range is $\{y|y \neq 1\}$.

Other orderings of the steps shown in Example 12 would also result in the graph of f. For example, try this one:

Step 1: $y = \frac{1}{x}$ Reciprocal function

Step 2: $y = \frac{1}{x-2}$ Replace x by x − 2; horizontal shift to the right 2 units.

Step 3: $y = \frac{3}{x-2}$ Multiply by 3; vertical stretch of the graph of $y = \frac{1}{x-2}$ by a factor of 3.

Step 4: $y = \frac{3}{x-2} + 1$ Add 1; vertical shift up 1 unit.

Hint: Although the order in which transformations are performed can be altered, you may consider using the following order for consistency:

1. Reflections
2. Compressions and stretches
3. Shifts

EXAMPLE 13 **Combining Graphing Procedures**

Graph the function $f(x) = \sqrt{1-x} + 2$. Find the domain and range of f.

Solution Because horizontal shifts require the form $x - h$, begin by rewriting $f(x)$ as $f(x) = \sqrt{1-x} + 2 = \sqrt{-(x-1)} + 2$. Now use the following steps:

Step 1: $y = \sqrt{x}$ *Square root function*

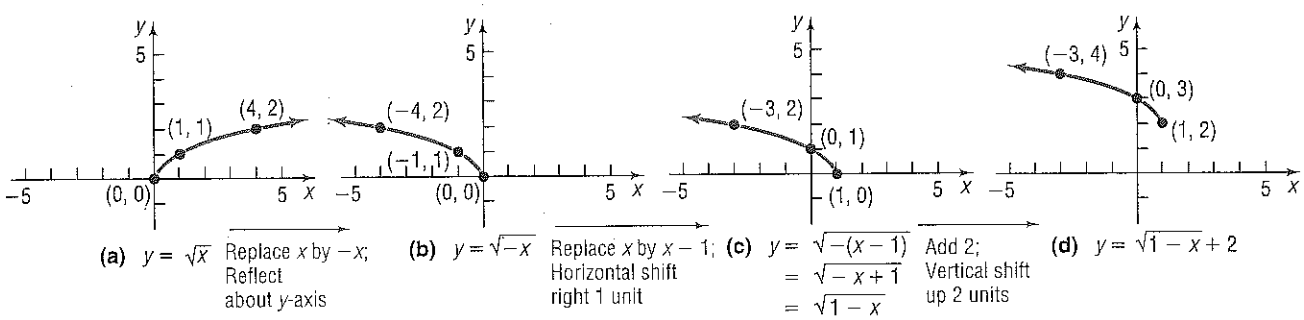
Step 2: $y = \sqrt{-x}$ *Replace x by $-x$; reflect about the y-axis.*

Step 3: $y = \sqrt{-(x-1)} = \sqrt{1-x}$ *Replace x by $x - 1$; horizontal shift to the right 1 unit.*

Step 4: $y = \sqrt{1-x} + 2$ *Add 2; vertical shift up 2 units.*

See Figure 57.

Figure 57



The domain of f is $(-\infty, 1]$ and the range is $[2, \infty)$.

Now Work PROBLEM 61

1.5 Assess Your Understanding

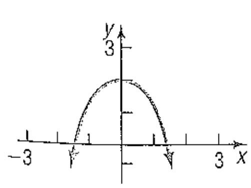
Concepts and Vocabulary

1. Suppose that the graph of a function f is known. Then the graph of $y = f(x-2)$ may be obtained by a(n) ____________ shift of the graph of f to the ____________ a distance of 2 units.
2. Suppose that the graph of a function f is known. Then the graph of $y = f(-x)$ may be obtained by a reflection about the ____________-axis of the graph of the function $y = f(x)$.
3. Suppose that the graph of a function g is known. The graph of $y = g(x) + 2$ may be obtained by a ____________ shift of the graph of g ____________ a distance of 2 units.
4. ***True or False*** The graph of $y = -f(x)$ is the reflection about the x-axis of the graph of $y = f(x)$.
5. ***True or False*** To obtain the graph of $f(x) = \sqrt{x+2}$, shift the graph of $y = \sqrt{x}$ horizontally to the right 2 units.
6. ***True or False*** To obtain the graph of $f(x) = x^3 + 5$, shift the graph of $y = x^3$ vertically up 5 units.

Skill Building

In Problems 7–18, match each graph to one of the following functions:

A. $y = x^2 + 2$	B. $y = -x^2 + 2$	C. $y = \lvert x\rvert + 2$	D. $y = -\lvert x\rvert + 2$
E. $y = (x-2)^2$	F. $y = -(x+2)^2$	G. $y = \lvert x-2\rvert$	H. $y = -\lvert x+2\rvert$
I. $y = 2x^2$	J. $y = -2x^2$	K. $y = 2\lvert x\rvert$	L. $y = -2\lvert x\rvert$

7.


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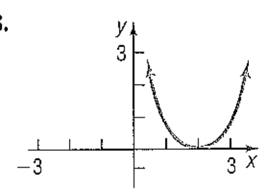

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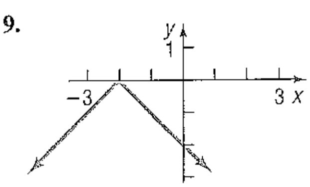

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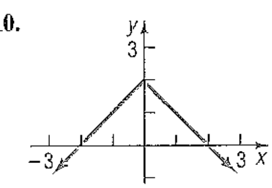

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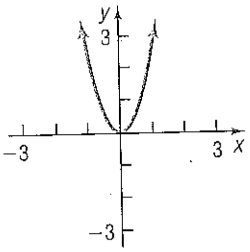

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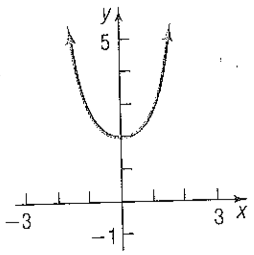

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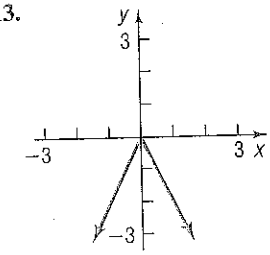

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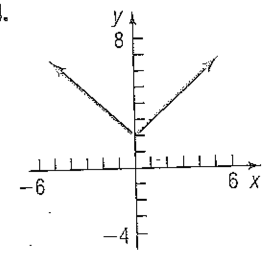

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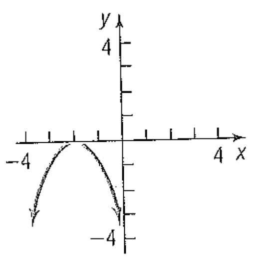

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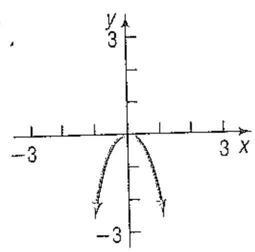

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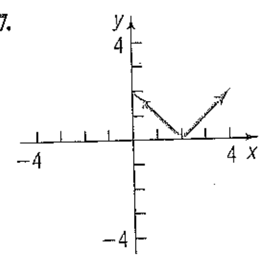
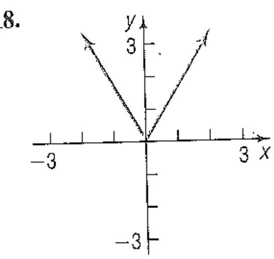


18.


In Problems 19–26, write the function whose graph is the graph of $y = x^3$, but is:

19. Shifted to the right 4 units

20. Shifted to the left 4 units

21. Shifted up 4 units

22. Shifted down 4 units

23. Reflected about the y-axis

24. Reflected about the x-axis

25. Vertically stretched by a factor of 4

26. Horizontally stretched by a factor of 4

In Problems 27–30, find the function that is finally graphed after each of the following transformations is applied to the graph of $y = \sqrt{x}$ in the order stated.

27. (1) Shift up 2 units
(2) Reflect about the x-axis
(3) Reflect about the y-axis

28. (1) Reflect about the x-axis
(2) Shift right 3 units
(3) Shift down 2 units

29. (1) Reflect about the x-axis
(2) Shift up 2 units
(3) Shift left 3 units

30. (1) Shift up 2 units
(2) Reflect about the y-axis
(3) Shift left 3 units

31. If $(3, 6)$ is a point on the graph of $y = f(x)$, which of the following points must be on the graph of $y = -f(x)$?
(a) $(6, 3)$ (b) $(6, -3)$
(c) $(3, -6)$ (d) $(-3, 6)$

32. If $(3, 6)$ is a point on the graph of $y = f(x)$, which of the following points must be on the graph of $y = f(-x)$?
(a) $(6, 3)$ (b) $(6, -3)$
(c) $(3, -6)$ (d) $(-3, 6)$

33. If $(1, 3)$ is a point on the graph of $y = f(x)$, which of the following points must be on the graph of $y = 2f(x)$?
(a) $(1, 3)$ (b) $(2, 3)$
(c) $(1, 6)$ (d) $\left(\frac{1}{2}, 3\right)$

34. If $(4, 2)$ is a point on the graph of $y = f(x)$, which of the following points must be on the graph of $y = f(2x)$?
(a) $(4, 1)$ (b) $(8, 2)$
(c) $(2, -2)$ (d) $(4, 4)$

35. Suppose that the x-intercepts of the graph of $y = f(x)$ are -5 and 3.
(a) What are the x-intercepts of the graph of $y = f(x + 2)$?
(b) What are the x-intercepts of the graph of $y = f(x - 2)$?
(c) What are the x-intercepts of the graph of $y = 4f(x)$?
(d) What are the x-intercepts of the graph of $y = f(-x)$?

36. Suppose that the x-intercepts of the graph of $y = f(x)$ are -8 and 1.
(a) What are the x-intercepts of the graph of $y = f(x + 4)$?
(b) What are the x-intercepts of the graph of $y = f(x - 3)$?
(c) What are the x-intercepts of the graph of $y = 2f(x)$?
(d) What are the x-intercepts of the graph of $y = f(-x)$?

37. Suppose that the function $y = f(x)$ is increasing on the interval $(-1, 5)$.
(a) Over what interval is the graph of $y = f(x + 2)$ increasing?
(b) Over what interval is the graph of $y = f(x - 5)$ increasing?
(c) What can be said about the graph of $y = -f(x)$?
(d) What can be said about the graph of $y = f(-x)$?

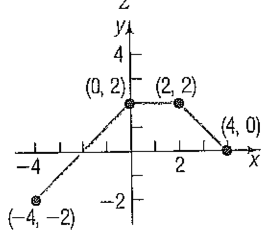
38. Suppose that the function $y = f(x)$ is decreasing on the interval $(-2, 7)$.
(a) Over what interval is the graph of $y = f(x + 2)$ decreasing?
(b) Over what interval is the graph of $y = f(x - 5)$ decreasing?
(c) What can be said about the graph of $y = -f(x)$?
(d) What can be said about the graph of $y = f(-x)$?

In Problems 39–68, graph each function using the techniques of shifting, compressing, stretching, and/or reflecting. Start with the graph of the basic function (for example, $y = x^2$) and show all stages. Be sure to show at least three key points. Find the domain and the range of each function.

39. $f(x) = x^2 - 1$ 40. $f(x) = x^2 + 4$ 41. $g(x) = x^3 + 1$
42. $g(x) = x^3 - 1$ 43. $h(x) = \sqrt{x + 2}$ 44. $h(x) = \sqrt{x} + 1$
45. $f(x) = (x - 1)^3 + 2$ 46. $f(x) = (x + 2)^3 - 3$ 47. $g(x) = 4\sqrt{x}$
48. $g(x) = \frac{1}{2}\sqrt{x}$ 49. $h(x) = \frac{1}{2x}$ 50. $h(x) = \sqrt[3]{2x}$
51. $f(x) = -\sqrt[3]{x}$ 52. $f(x) = -\sqrt{x}$ 53. $g(x) = \sqrt[3]{-x}$
54. $g(x) = \frac{1}{-x}$ 55. $h(x) = -x^3 + 2$ 56. $h(x) = \frac{1}{-x} + 2$
57. $f(x) = 2(x + 1)^2 - 3$ 58. $f(x) = 3(x - 2)^2 + 1$ 59. $g(x) = 2\sqrt{x - 2} + 1$
60. $g(x) = 3|x + 1| - 3$ 61. $h(x) = \sqrt{-x} - 2$ 62. $h(x) = \frac{4}{x} + 2$
63. $f(x) = -(x + 1)^3 - 1$ 64. $f(x) = -4\sqrt{x - 1}$ 65. $g(x) = 2|1 - x|$
66. $g(x) = 4\sqrt{2 - x}$ 67. $h(x) = 2\,\text{int}(x - 1)$ 68. $h(x) = \text{int}(-x)$

In Problems 69–72, the graph of a function f is illustrated. Use the graph of f as the first step toward graphing each of the following functions:

(a) $F(x) = f(x) + 3$ (b) $G(x) = f(x + 2)$ (c) $P(x) = -f(x)$ (d) $H(x) = f(x + 1) - 2$
(e) $Q(x) = \frac{1}{2}f(x)$ (f) $g(x) = f(-x)$ (g) $h(x) = f(2x)$

69.


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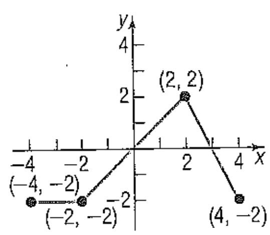

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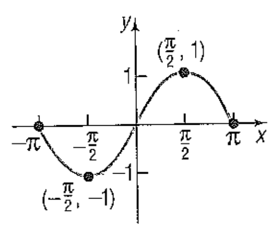

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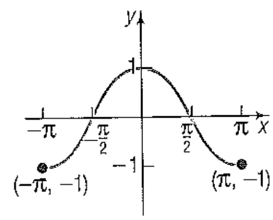


Mixed Practice

In Problems 73–82, complete the square of each quadratic expression. Then graph each function using the technique of shifting. (If necessary, refer to Appendix A, Section A.4 to review completing the square.)

73. $f(x) = x^2 + 2x$ 74. $f(x) = x^2 - 6x$ 75. $f(x) = x^2 - 8x + 1$ 76. $f(x) = x^2 + 4x + 2$
77. $f(x) = x^2 + x + 1$ 78. $f(x) = x^2 - x + 1$ 79. $f(x) = 2x^2 - 12x + 19$ 80. $f(x) = 3x^2 + 6x + 1$
81. $f(x) = -3x^2 - 12x - 17$ 82. $f(x) = -2x^2 - 12x - 13$

83. (a) Graph $f(x) = |x - 3| - 3$ using transformations.
(b) Find the area of the region bounded by f and the x-axis that lies below the x-axis.

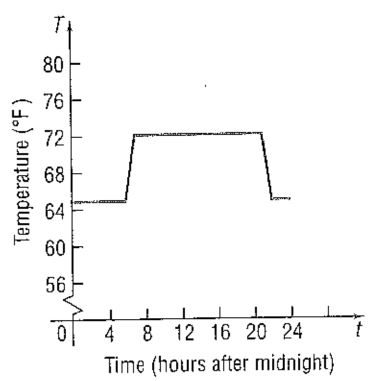
84. (a) Graph $f(x) = -2|x - 4| + 4$ using transformations.
(b) Find the area of the region bounded by f and the x-axis that lies above the x-axis.

Applications and Extensions

85. The equation $y = (x - c)^2$ defines a *family of parabolas*, one parabola for each value of c. On one set of coordinate axes, graph the members of the family for $c = 0$, $c = 3$, and $c = -2$.

86. Repeat Problem 85 for the family of parabolas $y = x^2 + c$.

87. **Thermostat Control** Energy conservation experts estimate that homeowners can save 5% to 10% on winter heating bills by programming their thermostats 5 to 10 degrees lower while sleeping. In the given graph, the temperature T (in degrees Fahrenheit) of a home is given as a function of time t (in hours after midnight) over a 24-hour period.



(a) At what temperature is the thermostat set during daytime hours? At what temperature is the thermostat set overnight?
(b) The homeowner reprograms the thermostat to $y = T(t) - 2$. Explain how this affects the temperature in the house. Graph this new function.
(c) The homeowner reprograms the thermostat to $y = T(t + 1)$. Explain how this affects the temperature in the house. Graph this new function.

Source: Roger Albright, 547 Ways to Be Fuel Smart, 2000

88. Digital Music Revenues The total projected worldwide digital music revenues R, in millions of dollars, for the years 2012 through 2017 can be estimated by the function

$$R(x) = 28.6x^2 + 300x + 4843$$

where x is the number of years after 2012.
(a) Find $R(0)$, $R(3)$, and $R(5)$ and explain what each value represents.
(b) Find $r = R(x - 2)$.
(c) Find $r(2)$, $r(5)$, and $r(7)$ and explain what each value represents.
(d) In the model r, what does x represent?
(e) Would there be an advantage in using the model r when estimating the projected revenues for a given year instead of the model R?

Source: IFPI Digital Music Report, 2013

89. Temperature Measurements The relationship between the Celsius (°C) and Fahrenheit (°F) scales for measuring temperature is given by the equation

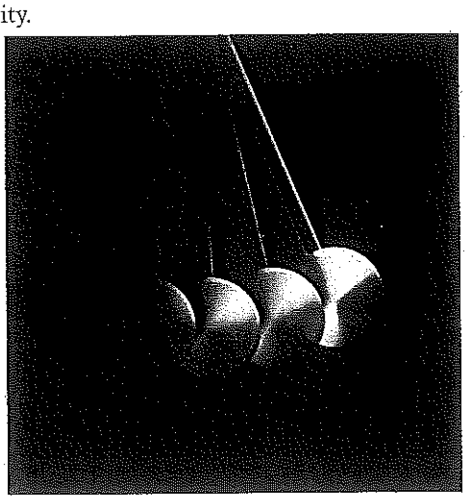
$$F = \frac{9}{5}C + 32$$

The relationship between the Celsius (°C) and Kelvin (K) scales is $K = C + 273$. Graph the equation $F = \frac{9}{5}C + 32$ using degrees Fahrenheit on the y-axis and degrees Celsius on the x-axis. Use the techniques introduced in this section to obtain the graph showing the relationship between Kelvin and Fahrenheit temperatures.

90. Period of a Pendulum The period T (in seconds) of a simple pendulum is a function of its length l (in feet) defined by the equation

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where $g \approx 32.2$ feet per second per second is the acceleration of gravity.



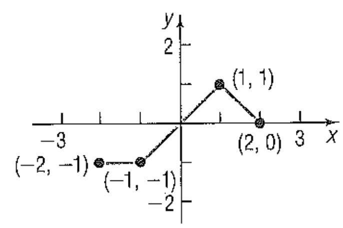
(a) Use a graphing utility to graph the function $T = T(l)$.
(b) Now graph the functions $T = T(l + 1)$, $T = T(l + 2)$, and $T = T(l + 3)$.
(c) Discuss how adding to the length l changes the period T.
(d) Now graph the functions $T = T(2l)$, $T = T(3l)$, and $T = T(4l)$.
(e) Discuss how multiplying the length l by factors of 2, 3, and 4 changes the period T.

91. Cigar Company Profits The daily profits of a cigar company from selling x cigars are given by

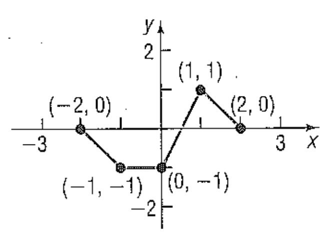
$$p(x) = -0.05x^2 + 100x - 2000$$

The government wishes to impose a tax on cigars (sometimes called a *sin tax*) that gives the company the option of either paying a flat tax of \$10,000 per day or a tax of 10% on profits. As chief financial officer (CFO) of the company, you need to decide which tax is the better option for the company.
(a) On the same screen, graph $Y_1 = p(x) - 10{,}000$ and $Y_2 = (1 - 0.10)p(x)$.
(b) Based on the graph, which option would you select? Why?
(c) Using the terminology learned in this section, describe each graph in terms of the graph of $p(x)$.
(d) Suppose that the government offered the options of a flat tax of \$4800 or a tax of 10% on profits. Which would you select? Why?

92. The graph of a function f is illustrated in the figure.
(a) Draw the graph of $y = |f(x)|$.
(b) Draw the graph of $y = f(|x|)$.



93. The graph of a function f is illustrated in the figure.
(a) Draw the graph of $y = |f(x)|$.
(b) Draw the graph of $y = f(|x|)$.



94. Suppose $(1, 3)$ is a point on the graph of $y = f(x)$.
(a) What point is on the graph of $y = f(x + 3) - 5$?
(b) What point is on the graph of $y = -2f(x - 2) + 1$?
(c) What point is on the graph of $y = f(2x + 3)$?

95. Suppose $(-3, 5)$ is a point on the graph of $y = g(x)$.
(a) What point is on the graph of $y = g(x + 1) - 3$?
(b) What point is on the graph of $y = -3g(x - 4) + 3$?
(c) What point is on the graph of $y = g(3x + 9)$?

Discussion and Writing

96. Suppose that the graph of a function f is known. Explain how the graph of $y = 4f(x)$ differs from the graph of $y = f(4x)$.

97. Suppose that the graph of a function f is known. Explain how the graph of $y = f(x) - 2$ differs from the graph of $y = f(x - 2)$.

98. The area under the curve $y = \sqrt{x}$ bounded below by the x-axis and on the right by $x = 4$ is $\frac{16}{3}$ square units. Using the ideas presented in this section, what do you think is the area under the curve of $y = \sqrt{-x}$ bounded below by the x-axis and on the left by $x = -4$? Justify your answer.

99. Explain how the range of the function $f(x) = x^2$ compares to the range of $g(x) = f(x) + k$.

100. Explain how the domain of $g(x) = \sqrt{x}$ compares to the domain of $g(x - k)$, where $k \geq 0$.

1.6 Mathematical Models: Building Functions

OBJECTIVE 1 Build and Analyze Functions (p. 101)

1 Build and Analyze Functions

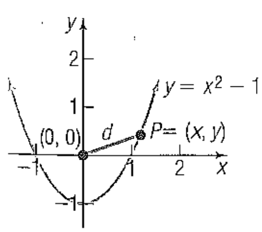
Real-world problems often result in mathematical models that involve functions. These functions need to be constructed or built based on the information given. In building functions, we must be able to translate the verbal description into the language of mathematics. This is done by assigning symbols to represent the independent and dependent variables and then finding the function or rule that relates these variables.

EXAMPLE 1 Finding the Distance from the Origin to a Point on a Graph

Let $P = (x, y)$ be a point on the graph of $y = x^2 - 1$.

(a) Express the distance d from P to the origin O as a function of x.
(b) What is d if $x = 0$?
(c) What is d if $x = 1$?
(d) What is d if $x = \frac{\sqrt{2}}{2}$?
(e) Use a graphing utility to graph the function $d = d(x), x \geq 0$. Rounded to two decimal places, find the value(s) of x at which d has a local minimum. [This gives the point(s) on the graph of $y = x^2 - 1$ closest to the origin.]

Figure 58



Solution

(a) Figure 58 illustrates the graph of $y = x^2 - 1$. The distance d from P to O is

$$d = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}$$

Since P is a point on the graph of $y = x^2 - 1$, substitute $x^2 - 1$ for y. Then

$$d(x) = \sqrt{x^2 + (x^2 - 1)^2} = \sqrt{x^4 - x^2 + 1}$$

The distance d is expressed as a function of x.

(b) If $x = 0$, the distance d is

$$d(0) = \sqrt{0^4 - 0^2 + 1} = \sqrt{1} = 1$$

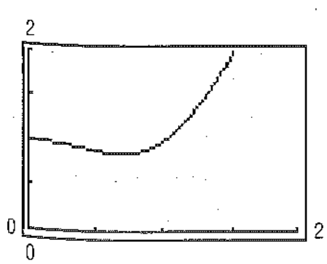
(c) If $x = 1$, the distance d is

$$d(1) = \sqrt{1^4 - 1^2 + 1} = 1$$

(d) If $x = \frac{\sqrt{2}}{2}$, the distance d is

$$d\left(\frac{\sqrt{2}}{2}\right) = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^4 - \left(\frac{\sqrt{2}}{2}\right)^2 + 1} = \sqrt{\frac{1}{4} - \frac{1}{2} + 1} = \frac{\sqrt{3}}{2}$$

Figure 59



(e) Figure 59 shows the graph of $Y_1 = \sqrt{x^4 - x^2 + 1}$. Using the MINIMUM feature on a graphing utility, we find that when $x \approx 0.71$, the value of d is smallest. The local minimum is $d \approx 0.87$ rounded to two decimal places. Since $d(x)$ is even, it follows by symmetry that when $x \approx -0.71$, the value of d is also a local minimum. Since $(\pm 0.71)^2 - 1 \approx -0.50$, the points $(-0.71, -0.50)$ and $(0.71, -0.50)$ on the graph of $y = x^2 - 1$ are closest to the origin.

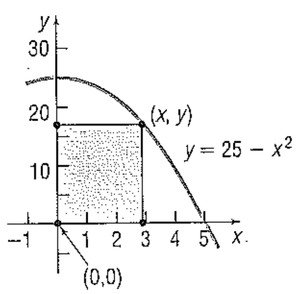
Now Work PROBLEM 1

EXAMPLE 2 Area of a Rectangle

A rectangle has one corner in quadrant I on the graph of $y = 25 - x^2$, another at the origin, a third on the positive y-axis, and the fourth on the positive x-axis. See Figure 60.

(a) Express the area A of the rectangle as a function of x.
(b) What is the domain of A?
(c) Graph $A = A(x)$.
(d) For what value of x is the area largest?

Figure 60



Solution

(a) The area A of the rectangle is $A = xy$, where $y = 25 - x^2$. Substituting this expression for y, we obtain $A(x) = x(25 - x^2) = 25x - x^3$.
(b) Since (x, y) is in quadrant I, we have $x > 0$. Also, $y = 25 - x^2 > 0$, which implies that $x^2 < 25$, so $-5 < x < 5$. Combining these restrictions, we have the domain of A as $\{x | 0 < x < 5\}$, or $(0, 5)$ using interval notation.
(c) See Figure 61 for the graph of $A = A(x)$.
(d) Using MAXIMUM, we find that the maximum area is 48.11 square units at $x = 2.89$ units, each rounded to two decimal places. See Figure 62.

Figure 61

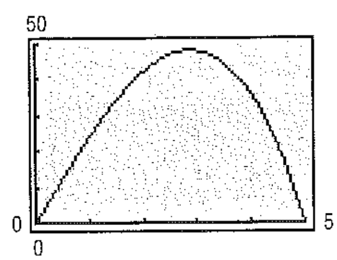
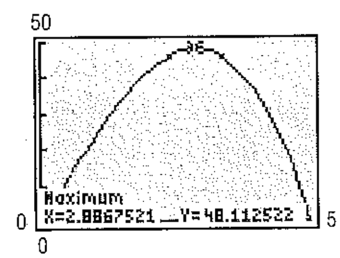


Figure 62



Now Work PROBLEM 7

EXAMPLE 3 Close Call?

Suppose two planes flying at the same altitude are headed toward each other. One plane is flying due south at a groundspeed of 400 miles per hour and is 600 miles from the potential intersection point of the planes. The other plane is flying due west with a groundspeed of 250 miles per hour and is 400 miles from the potential intersection point of the planes. See Figure 63.

(a) Build a model that expresses the distance d between the planes as a function of time t.
(b) Use a graphing utility to graph $d = d(t)$. How close do the planes come to each other? At what time are the planes closest?

Solution

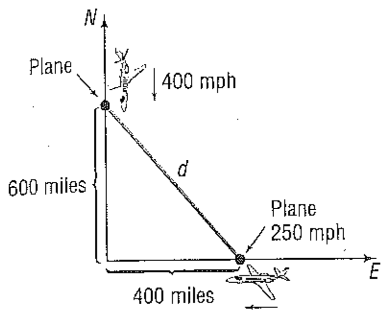
(a) Refer to Figure 63. The distance d between the two planes is the hypotenuse of a right triangle. At any time t the length of the north/south leg of the triangle is $600 - 400t$. At any time t, the length of the east/west leg of the triangle is $400 - 250t$. Using the Pythagorean Theorem, the square of the distance between the two planes is

$$d^2 = (600 - 400t)^2 + (400 - 250t)^2$$

Therefore, the distance between the two planes as a function of time is given by the model

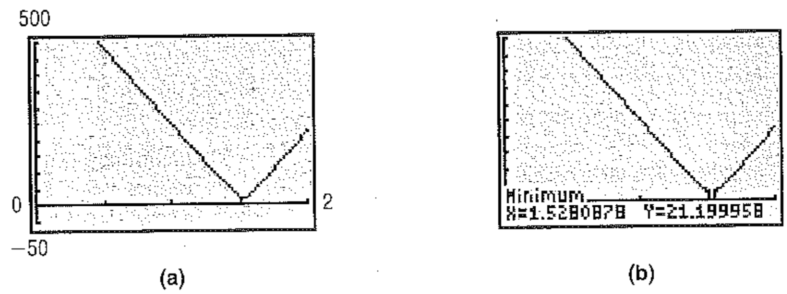
$$d(t) = \sqrt{(600 - 400t)^2 + (400 - 250t)^2}$$

Figure 63



(b) Figure 64(a) shows the graph of $d = d(t)$. Using MINIMUM, the minimum distance between the planes is 21.20 miles, and the time at which the planes are closest is after 1.53 hours, each rounded to two decimal places. See Figure 64(b).

Figure 64



Now Work PROBLEM 19

1.6 Assess Your Understanding

Applications and Extensions

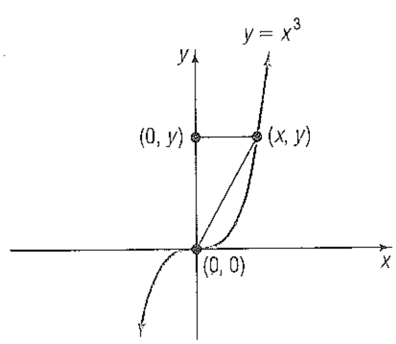
1. Let $P = (x, y)$ be a point on the graph of $y = x^2 - 8$.
   (a) Express the distance d from P to the origin as a function of x.
   (b) What is d if $x = 0$?
   (c) What is d if $x = 1$?
   (d) Use a graphing utility to graph $d = d(x)$.
   (e) For what values of x is d smallest?

2. Let $P = (x, y)$ be a point on the graph of $y = x^2 - 8$.
   (a) Express the distance d from P to the point $(0, -1)$ as a function of x.
   (b) What is d if $x = 0$?
   (c) What is d if $x = -1$?
   (d) Use a graphing utility to graph $d = d(x)$.
   (e) For what values of x is d smallest?

3. Let $P = (x, y)$ be a point on the graph of $y = \sqrt{x}$.
   (a) Express the distance d from P to the point $(1, 0)$ as a function of x.
   (b) Use a graphing utility to graph $d = d(x)$.
   (c) For what values of x is d smallest?

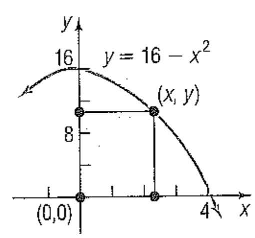
4. Let $P = (x, y)$ be a point on the graph of $y = \frac{1}{x}$.
   (a) Express the distance d from P to the origin as a function of x.
   (b) Use a graphing utility to graph $d = d(x)$.
   (c) For what values of x is d smallest?

5. A right triangle has one vertex on the graph of $y = x^3, x > 0$, at (x, y), another at the origin, and the third on the positive y-axis at $(0, y)$, as shown in the figure. Express the area A of the triangle as a function of x.

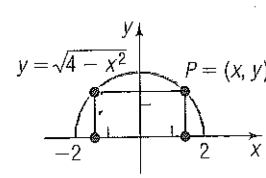


6. A right triangle has one vertex on the graph of $y = 9 - x^2, x > 0$, at (x, y), another at the origin, and the third on the positive x-axis at $(x, 0)$. Express the area A of the triangle as a function of x.

7. A rectangle has one corner in quadrant I on the graph of $y = 16 - x^2$, another at the origin, a third on the positive y-axis, and the fourth on the positive x-axis. See the figure.
   (a) Express the area A of the rectangle as a function of x.
   (b) What is the domain of A?
   (c) Graph $A = A(x)$. For what value of x is A largest?

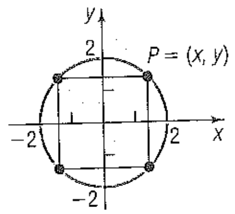


8. A rectangle is inscribed in a semicircle of radius 2. See the figure. Let $P = (x, y)$ be the point in quadrant I that is a vertex of the rectangle and is on the circle.



   (a) Express the area A of the rectangle as a function of x.
   (b) Express the perimeter p of the rectangle as a function of x.
   (c) Graph $A = A(x)$. For what value of x is A largest?
   (d) Graph $p = p(x)$. For what value of x is p largest?

9. A rectangle is inscribed in a circle of radius 2. See the figure on the next page. Let $P = (x, y)$ be the point in quadrant I that is a vertex of the rectangle and is on the circle.
   (a) Express the area A of the rectangle as a function of x.



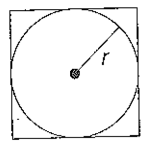
$x^2 + y^2 = 4$

(b) Express the perimeter p of the rectangle as a function of x.

(c) Graph $A = A(x)$. For what value of x is A largest?

(d) Graph $p = p(x)$. For what value of x is p largest?

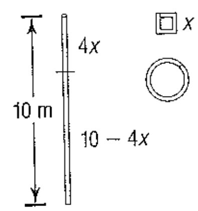
10. A circle of radius r is inscribed in a square. See the figure.



(a) Express the area A of the square as a function of the radius r of the circle.

(b) Express the perimeter p of the square as a function of r.

11. Geometry A wire 10 meters long is to be cut into two pieces. One piece will be shaped as a square, and the other piece will be shaped as a circle. See the figure.



(a) Express the total area A enclosed by the pieces of wire as a function of the length x of a side of the square.

(b) What is the domain of A?

(c) Graph $A = A(x)$. For what value of x is A smallest?

12. Geometry A wire 10 meters long is to be cut into two pieces. One piece will be shaped as an equilateral triangle, and the other piece will be shaped as a circle.

(a) Express the total area A enclosed by the pieces of wire as a function of the length x of a side of the equilateral triangle.

(b) What is the domain of A?

(c) Graph $A = A(x)$. For what value of x is A smallest?

13. A wire of length x is bent into the shape of a circle.

(a) Express the circumference C of the circle as a function of x.

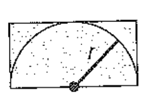
(b) Express the area A of the circle as a function of x.

14. A wire of length x is bent into the shape of a square.

(a) Express the perimeter p of the square as a function of x.

(b) Express the area A of the square as a function of x.

15. Geometry A semicircle of radius r is inscribed in a rectangle so that the diameter of the semicircle is the length of the rectangle. See the figure.

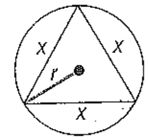


(a) Express the area A of the rectangle as a function of the radius r of the semicircle.

(b) Express the perimeter p of the rectangle as a function of r.

16. Geometry An equilateral triangle is inscribed in a circle of radius r. See the figure. Express the circumference C of the circle as a function of the length x of a side of the triangle.

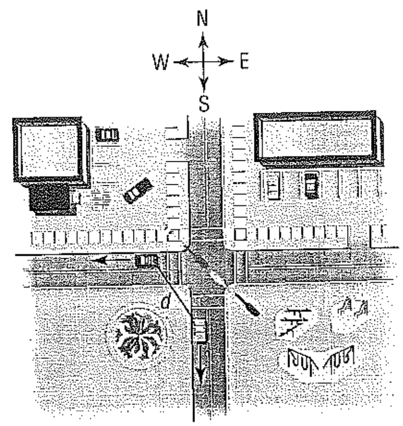
[**Hint:** First show that $r^2 = \dfrac{x^2}{3}$.]



17. Geometry An equilateral triangle is inscribed in a circle of radius r. See the figure in Problem 16. Express the area A within the circle, but outside the triangle, as a function of the length x of a side of the triangle.

18. Uniform Motion Two cars leave an intersection at the same time. One is headed south at a constant speed of 30 miles per hour, and the other is headed west at a constant speed of 40 miles per hour (see the figure). Build a model that expresses the distance d between the cars as a function of the time t.

[**Hint:** At $t = 0$, the cars leave the intersection.]



19. Uniform Motion Two cars are approaching an intersection. One is 2 miles south of the intersection and is moving at a constant speed of 30 miles per hour. At the same time, the other car is 3 miles east of the intersection and is moving at a constant speed of 40 miles per hour.

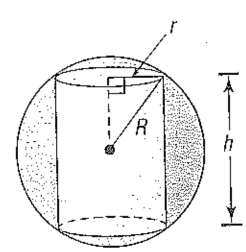
(a) Build a model that expresses the distance d between the cars as a function of time t.

[**Hint:** At $t = 0$, the cars are 2 miles south and 3 miles east of the intersection, respectively.]

(b) Use a graphing utility to graph $d = d(t)$. For what value of t is d smallest?

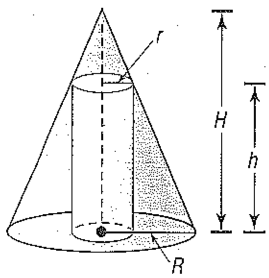
20. Inscribing a Cylinder in a Sphere Inscribe a right circular cylinder of height h and radius r in a sphere of fixed radius R. See the illustration. Express the volume V of the cylinder as a function of h.

[**Hint:** $V = \pi r^2 h$. Note also the right triangle.]



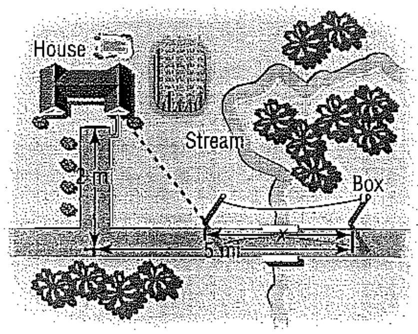
Sphere

21. **Inscribing a Cylinder in a Cone** Inscribe a right circular cylinder of height h and radius r in a cone of fixed radius R and fixed height H. See the illustration. Express the volume V of the cylinder as a function of r.
[**Hint:** $V = \pi r^2 h$. Note also the similar triangles.]



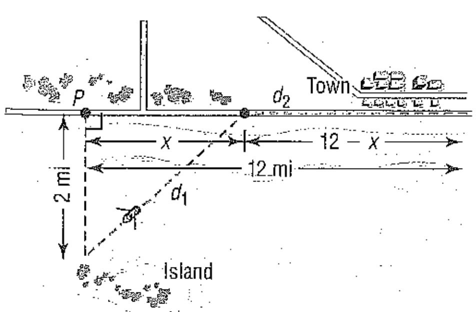
Cone

22. **Installing Cable TV** MetroMedia Cable is asked to provide service to a customer whose house is located 2 miles from the road along which the cable is buried. The nearest connection box for the cable is located 5 miles down the road. See the figure.



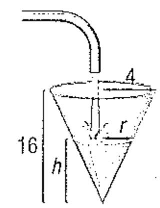
(a) If the installation cost is $500 per mile along the road and $700 per mile off the road, build a model that expresses the total cost C of installation as a function of the distance x (in miles) from the connection box to the point where the cable installation turns off the road. Give the domain.
(b) Compute the cost if $x = 1$ mile.
(c) Compute the cost if $x = 3$ miles.
(d) Graph the function $C = C(x)$. Use TRACE to see how the cost C varies as x changes from 0 to 5.
(e) What value of x results in the least cost?

23. **Time Required to Go from an Island to a Town** An island is 2 miles from the nearest point P on a straight shoreline. A town is 12 miles down the shore from P. See the illustration.

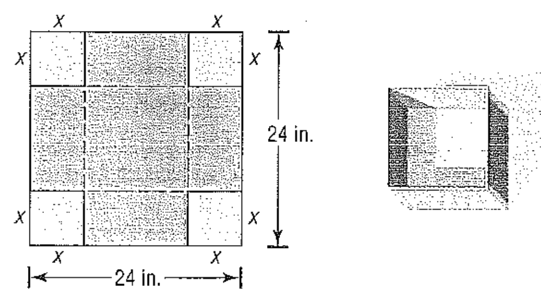


(a) If a person can row a boat at an average speed of 3 miles per hour and the same person can walk 5 miles per hour, build a model that expresses the time T that it takes to go from the island to town as a function of the distance x from P to where the person lands the boat.
(b) What is the domain of T?
(c) How long will it take to travel from the island to town if the person lands the boat 4 miles from P?
(d) How long will it take if the person lands the boat 8 miles from P?

24. **Filling a Conical Tank** Water is poured into a container in the shape of a right circular cone with radius 4 feet and height 16 feet. See the figure. Express the volume V of the water in the cone as a function of the height h of the water.
[**Hint:** The volume V of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$.]



25. **Constructing an Open Box** An open box with a square base is to be made from a square piece of cardboard 24 inches on a side by cutting out a square from each corner and turning up the sides. See the figure.



(a) Express the volume V of the box as a function of the length x of the side of the square cut from each corner.
(b) What is the volume if a 3-inch square is cut out?
(c) What is the volume if a 10-inch square is cut out?
(d) Graph $V = V(x)$. For what value of x is V largest?

26. **Constructing an Open Box** An open box with a square base is required to have a volume of 10 cubic feet.
(a) Express the amount A of material used to make such a box as a function of the length x of a side of the square base.
(b) How much material is required for a base 1 foot by 1 foot?
(c) How much material is required for a base 2 feet by 2 feet?
(d) Use a graphing utility to graph $A = A(x)$. For what value of x is A smallest?

1.7 Building Mathematical Models Using Variation

OBJECTIVES
1 Construct a Model Using Direct Variation (p. 106)
2 Construct a Model Using Inverse Variation (p. 107)
3 Construct a Model Using Joint or Combined Variation (p. 107)

When a mathematical model is developed for a real-world problem, it often involves relationships between quantities that are expressed in terms of proportionality:

Force is proportional to acceleration.

When an ideal gas is held at a constant temperature, pressure and volume are inversely proportional.

The force of attraction between two objects is inversely proportional to the square of the distance between them.

Revenue is directly proportional to sales.

Each of these statements illustrates the idea of **variation,** or how one quantity varies in relation to another quantity. Quantities may vary *directly*, *inversely*, or *jointly*.

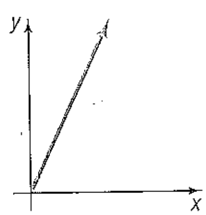
1 Construct a Model Using Direct Variation

DEFINITION Let x and y denote two quantities. Then y **varies directly** with x, or y is **directly proportional to** x, if there is a nonzero number k such that

$$y = kx$$

The number k is called the **constant of proportionality.**

Figure 65



The graph in Figure 65 illustrates the relationship between y and x if y varies directly with x and $k > 0, x \geq 0$. Note that the constant of proportionality is, in fact, the slope of the line.

If two quantities vary directly, then knowing the value of each quantity in one instance enables us to write a model that is true in all cases.

EXAMPLE 1 Mortgage Payments

The monthly payment p on a mortgage varies directly with the amount B borrowed. If the monthly payment on a 30-year mortgage is \$6.65 for every \$1000 borrowed, find a model that relates the monthly payment p to the amount B borrowed for a mortgage with these terms. Then find the monthly payment p when the amount B borrowed is \$120,000.

Solution Because p varies directly with B, we know that

$$p = kB$$

for some constant k. Because $p = 6.65$ when $B = 1000$, it follows that

$$6.65 = k(1000)$$

$$k = 0.00665 \quad \text{Solve for } k.$$

Since $p = kB$, we have

$$p = 0.00665B \quad \text{The model}$$

In particular, when $B = \$120{,}000$, we find that

$$p = 0.00665(\$120{,}000) = \$798$$

Figure 66

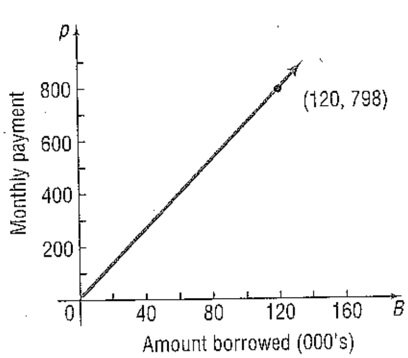


Figure 66 illustrates the relationship between the monthly payment p and the amount B borrowed.

Now Work PROBLEMS 3 AND 21

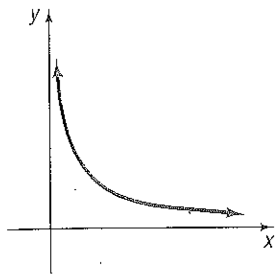
2 Construct a Model Using Inverse Variation

DEFINITION

Let x and y denote two quantities. Then y **varies inversely** with x, or y is **inversely proportional to** x, if there is a nonzero constant k such that

$$y = \frac{k}{x}$$

Figure 67

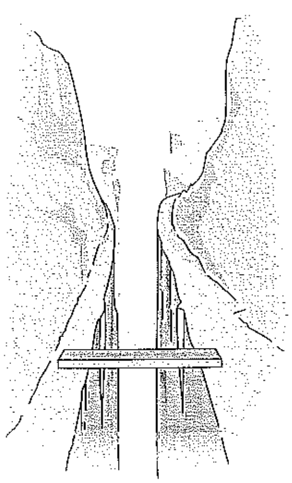


The graph in Figure 67 illustrates the relationship between y and x if y varies inversely with x and $k > 0, x > 0$.

EXAMPLE 2 **Maximum Weight That Can Be Supported by a Piece of Pine**

See Figure 68. The maximum weight W that can be safely supported by a 2-inch by 4-inch piece of pine varies inversely with its length l. Experiments indicate that the maximum weight that a 10-foot-long 2-by-4 piece of pine can support is 500 pounds. Write a model relating the maximum weight W (in pounds) to length l (in feet). Find the maximum weight W that can be safely supported by a length of 25 feet.

Figure 68



Solution Because W varies inversely with l, we know that

$$W = \frac{k}{l}$$

for some constant k. Because $W = 500$ when $l = 10$, we have

$$500 = \frac{k}{10}$$
$$k = 5000$$

Since $W = \frac{k}{l}$, we have

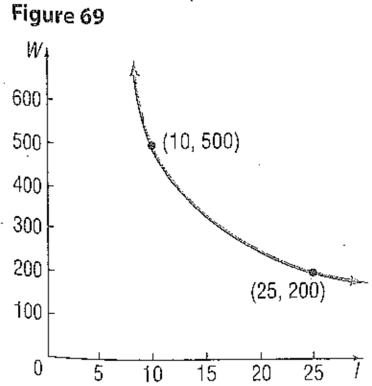
$$W = \frac{5000}{l} \quad \text{The Model}$$

In particular, the maximum weight W that can be safely supported by a piece of pine 25 feet in length is

$$W = \frac{5000}{25} = 200 \text{ pounds}$$

Figure 69 illustrates the relationship between the weight W and the length l.

Figure 69



Now Work PROBLEM 31

3 Construct a Model Using Joint or Combined Variation

When a variable quantity Q is proportional to the product of two or more other variables, we say that Q **varies jointly** with these quantities. Combinations of direct and/or inverse variation may also occur. This is usually referred to as **combined variation.**

Let's look at an example.

EXAMPLE 3 **Loss of Heat through a Wall**

The loss of heat through a wall varies jointly with the area of the wall and the difference between the inside and outside temperatures, and inversely with the thickness of the wall. Write an equation that relates these quantities.

Solution Begin by assigning symbols to represent the quantities:

L = Heat loss T = Temperature difference
A = Area of wall d = Thickness of wall

Then

$$L = k\frac{AT}{d}$$

where k is the constant of proportionality.

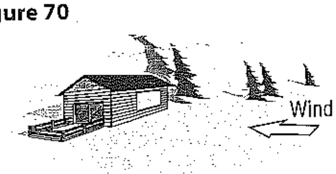
In direct or inverse variation, the quantities that vary may be raised to powers. For example, in the early seventeenth century, Johannes Kepler (1571–1630) discovered that the square of the period of revolution T of a planet around the Sun varies directly with the cube of its mean distance a from the Sun. That is, $T^2 = ka^3$, where k is the constant of proportionality.

EXAMPLE 4 **Force of the Wind on a Window**

The force F of the wind on a flat surface positioned at a right angle to the direction of the wind varies jointly with the area A of the surface and the square of the speed v of the wind. A wind of 30 miles per hour blowing on a window measuring 4 feet by 5 feet has a force of 150 pounds. See Figure 70. What force does a wind of 50 miles per hour cause on a window measuring 3 feet by 4 feet?

Solution Since F varies jointly with A and v^2, we have

Figure 70



$$F = kAv^2$$

where k is the constant of proportionality. We are told that $F = 150$ pounds when $A = 4 \cdot 5 = 20$ feet^2 and $v = 30$ miles per hour. This gives

$$150 = k(20)(900) \quad F = kAv^2, F = 150, A = 20, v = 30$$

$$k = \frac{1}{120}$$

Since $F = kAv^2$, we have

$$F = \frac{1}{120}Av^2$$

For a wind of 50 miles per hour blowing on a window whose area is $A = 3 \cdot 4 = 12$ square feet, the force F is

$$F = \frac{1}{120}(12)(2500) = 250 \text{ pounds}$$

Now Work PROBLEM 39

1.7 Assess Your Understanding

Concepts and Vocabulary

1. If x and y are two quantities, then y is directly proportional to x if there is a nonzero number k such that ______.

2. ***True or False*** If y varies directly with x, then $y = \frac{k}{x}$, where k is a constant.

Skill Building

In Problems 3–14, write a general formula to describe each variation.

3. y varies directly with x; $y = 2$ when $x = 10$

4. v varies directly with t; $v = 16$ when $t = 2$

5. A varies directly with x^2; $A = 4\pi$ when $x = 2$

6. V varies directly with x^3; $V = 36\pi$ when $x = 3$

7. F varies inversely with d^2; $F = 10$ when $d = 5$

8. y varies inversely with $\sqrt{x}$; $y = 4$ when $x = 9$

9. z varies directly with the sum of the squares of x and y; $z = 5$ when $x = 3$ and $y = 4$

10. T varies jointly with the cube root of x and the square of d; $T = 18$ when $x = 8$ and $d = 3$

11. M varies directly with the square of d and inversely with the square root of x; $M = 24$ when $x = 9$ and $d = 4$

12. z varies directly with the sum of the cube of x and the square of y; $z = 1$ when $x = 2$ and $y = 3$

13. The square of T varies directly with the cube of a and inversely with the square of d; $T = 2$ when $a = 2$ and $d = 4$

14. The cube of z varies directly with the sum of the squares of x and y; $z = 2$ when $x = 9$ and $y = 4$

Applications and Extensions

In Problems 15–20, write an equation that relates the quantities.

15. **Geometry** The volume V of a sphere varies directly with the cube of its radius r. The constant of proportionality is $\frac{4\pi}{3}$.

16. **Geometry** The square of the length of the hypotenuse c of a right triangle varies jointly with the sum of the squares of the lengths of its legs a and b. The constant of proportionality is 1.

17. **Geometry** The area A of a triangle varies jointly with the lengths of the base b and the height h. The constant of proportionality is $\frac{1}{2}$.

18. **Geometry** The perimeter p of a rectangle varies jointly with the sum of the lengths of its sides l and w. The constant of proportionality is 2.

19. **Physics: Newton's Law** The force F (in newtons) of attraction between two objects varies jointly with their masses m and M (in kilograms) and inversely with the square of the distance d (in meters) between them. The constant of proportionality is $G = 6.67 \times 10^{-11}$.

20. **Physics: Simple Pendulum** The **period** of a pendulum is the time required for one oscillation; the pendulum is usually referred to as **simple** when the angle made to the vertical is less than 5°. The period T of a simple pendulum (in seconds) varies directly with the square root of its length l (in feet). The constant of proportionality is $\frac{2\pi}{\sqrt{32}}$.

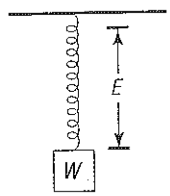
21. **Mortgage Payments** The monthly payment p on a mortgage varies directly with the amount B borrowed. If the monthly payment on a 30-year mortgage is \$6.49 for every \$1000 borrowed, find a model that relates the monthly payment p to the amount B borrowed for a mortgage with the same terms. Then find the monthly payment p when the amount B borrowed is \$145,000.

22. **Mortgage Payments** The monthly payment p on a mortgage varies directly with the amount B borrowed. If the monthly payment on a 15-year mortgage is \$8.99 for every \$1000 borrowed, find a model that relates the monthly payment p to the amount B borrowed for a mortgage with the same terms. Then find the monthly payment p when the amount B borrowed is \$175,000.

23. **Physics: Falling Objects** The distance s that an object falls is directly proportional to the square of the time t of the fall. If an object falls 16 feet in 1 second, how far will it fall in 3 seconds? How long will it take an object to fall 64 feet?

24. **Physics: Falling Objects** The velocity v of a falling object is directly proportional to the time t of the fall. If, after 2 seconds, the velocity of the object is 64 feet per second, what will its velocity be after 3 seconds?

25. Physics: Stretching a Spring The elongation E of a spring balance varies directly with the applied weight W (see the figure). If $E = 3$ when $W = 20$, find E when $W = 15$.



26. Physics: Vibrating String The rate of vibration of a string under constant tension varies inversely with the length of the string. If a string is 48 inches long and vibrates 256 times per second, what is the length of a string that vibrates 576 times per second?

27. Revenue Equation At the corner Shell station, the revenue R varies directly with the number g of gallons of gasoline sold. If the revenue is \$47.40 when the number of gallons sold is 12, find a model that relates revenue R to the number g of gallons of gasoline sold. Then find the revenue R when the number of gallons of gasoline sold is 10.5.

28. Cost Equation The cost C of chocolate-covered almonds varies directly with the number A of pounds of almonds purchased. If the cost is \$23.75 when the number of pounds of chocolate-covered almonds purchased is 5, find a model that relates the cost C to the number A of pounds of almonds purchased. Then find the cost C when the number of pounds of almonds purchased is 3.5.

29. Demand Suppose that the demand D for candy at the movie theater is inversely related to the price p.
(a) When the price of candy is \$2.75 per bag, the theater sells 156 bags of candy. Express the demand for candy in terms of its price.
(b) Determine the number of bags of candy that will be sold if the price is raised to \$3 a bag.

30. Driving to School The time t that it takes to get to school varies inversely with your average speed s.
(a) Suppose that it takes you 40 minutes to get to school when your average speed is 30 miles per hour. Express the driving time to school in terms of average speed.
(b) Suppose that your average speed to school is 40 miles per hour. How long will it take you to get to school?

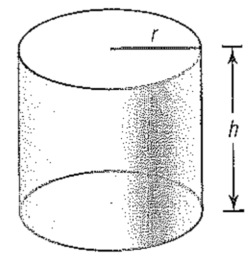
31. Pressure The volume V of a gas held at a constant temperature in a closed container varies inversely with its pressure P. If the volume of a gas is 600 cubic centimeters (cm^3) when the pressure is 150 millimeters of mercury (mm Hg), find the volume when the pressure is 200 mm Hg.

32. Resistance The current I in a circuit is inversely proportional to its resistance Z measured in ohms. Suppose that when the current in a circuit is 30 amperes, the resistance is 8 ohms. Find the current in the same circuit when the resistance is 10 ohms.

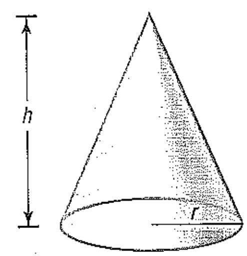
33. Weight The weight of an object above the surface of Earth varies inversely with the square of the distance from the center of Earth. If Maria weighs 125 pounds when she is on the surface of Earth (3960 miles from the center), determine Maria's weight when she is at the top of Mount McKinley (3.8 miles from the surface of Earth).

34. Intensity of Light The intensity I of light (measured in foot-candles) varies inversely with the square of the distance from the bulb. Suppose that the intensity of a 100-watt light bulb at a distance of 2 meters is 0.075 foot-candle. Determine the intensity of the bulb at a distance of 5 meters.

35. Geometry The volume V of a right circular cylinder varies jointly with the square of its radius r and its height h. The constant of proportionality is π. See the figure. Write an equation for V.



36. Geometry The volume V of a right circular cone varies jointly with the square of its radius r and its height h. The constant of proportionality is $\frac{\pi}{3}$. See the figure. Write an equation for V.



37. Weight of a Body The weight of a body above the surface of Earth varies inversely with the square of the distance from the center of Earth. If a certain body weighs 55 pounds when it is 3960 miles from the center of Earth, how much will it weigh when it is 3965 miles from the center?

38. Force of the Wind on a Window The force exerted by the wind on a plane surface varies jointly with the area of the surface and the square of the velocity of the wind. If the force on an area of 20 square feet is 11 pounds when the wind velocity is 22 miles per hour, find the force on a surface area of 47.125 square feet when the wind velocity is 36.5 miles per hour.

39. Horsepower The horsepower (hp) that a shaft can safely transmit varies jointly with its speed (in revolutions per minute, rpm) and the cube of its diameter. If a shaft of a certain material 2 inches in diameter can transmit 36 hp at 75 rpm, what diameter must the shaft have in order to transmit 45 hp at 125 rpm?

40. Chemistry: Gas Laws The volume V of an ideal gas varies directly with the temperature T and inversely with the pressure P. Write a model relating V, T, and P using k as the constant of proportionality. If a cylinder contains oxygen at a temperature of 300 K and a pressure of 15 atmospheres in a volume of 100 liters, what is the constant of proportionality k? If a piston is lowered into the cylinder, decreasing the volume occupied by the gas to 80 liters and raising the temperature to 310 K, what is the gas pressure?

41. **Physics: Kinetic Energy** The kinetic energy K of a moving object varies jointly with its mass m and the square of its velocity v. If an object weighing 25 kilograms and moving with a velocity of 10 meters per second has a kinetic energy of 1250 joules, find its kinetic energy when the velocity is 15 meters per second.

42. **Electrical Resistance of a Wire** The electrical resistance of a wire varies directly with the length of the wire and inversely with the square of the diameter of the wire. If a wire 432 feet long and 4 millimeters in diameter has a resistance of 1.24 ohms, find the length of a wire of the same material whose resistance is 1.44 ohms and whose diameter is 3 millimeters.

43. **Measuring the Stress of Materials** The stress in the material of a pipe subject to internal pressure varies jointly with the internal pressure and the internal diameter of the pipe and inversely with the thickness of the pipe. The stress is 100 pounds per square inch (ψ) when the diameter is 5 inches, the thickness is 0.75 inch, and the internal pressure is 25 pounds per square inch. Find the stress when the internal pressure is 40 pounds per square inch if the diameter is 8 inches and the thickness is 0.50 inch.

44. **Safe Load for a Beam** The maximum safe load for a horizontal rectangular beam varies jointly with the width of the beam and the square of the thickness of the beam and inversely with its length. If an 8-foot beam will support up to 750 pounds when the beam is 4 inches wide and 2 inches thick, what is the maximum safe load in a similar beam 10 feet long, 6 inches wide, and 2 inches thick?

Discussion and Writing

45. In the early seventeenth century, Johannes Kepler discovered that the square of the period T for a planet to orbit the Sun varies directly with the cube of its mean distance a from the Sun. Go to the library and research this law and Kepler's other two laws. Write a brief paper about these laws and Kepler's place in history.

46. Using a situation that has not been discussed in the text, write a real-world problem that you think involves two variables that vary directly. Exchange your problem with another student to solve and critique.

47. Using a situation that has not been discussed in the text, write a real-world problem that you think involves two variables that vary inversely. Exchange your problem with another student to solve and critique.

48. Using a situation that has not been discussed in the text, write a real-world problem that you think involves three variables that vary jointly. Exchange your problem with another student to solve and critique.

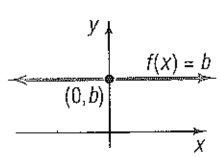
Chapter Review

Library of Functions

Constant function (p. 80)

$f(x) = b$

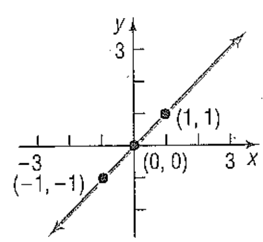
The graph is a horizontal line with y-intercept b.



Identity function (p. 80)

$f(x) = x$

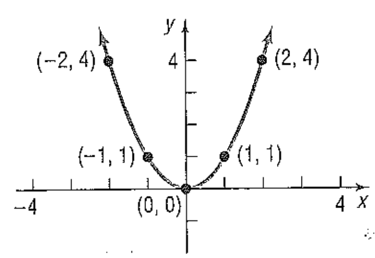
The graph is a line with slope 1 and y-intercept 0.



Square function (pp. 80–81)

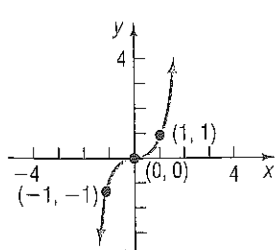
$f(x) = x^2$

The graph is a parabola with intercept at $(0, 0)$.



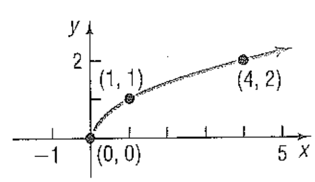
Cube function (p. 81)

$f(x) = x^3$



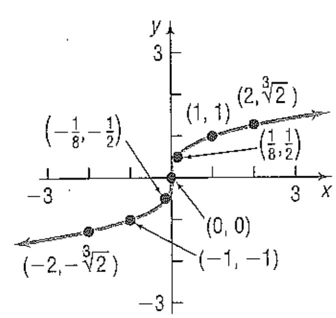
Square root function (pp. 78 and 81)

$f(x) = \sqrt{x}$



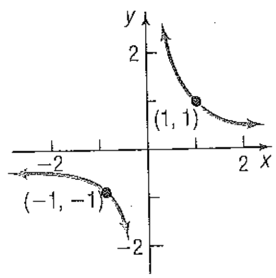
Cube root function (pp. 79 and 81)

$f(x) = \sqrt[3]{x}$



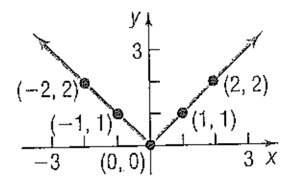
Reciprocal function (p. 81)

$f(x) = \frac{1}{x}$



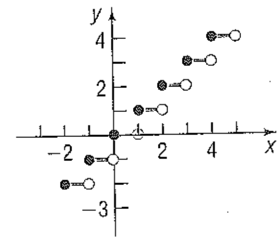
Absolute value function (pp. 80 and 82)

$f(x) = |x|$



Greatest integer function (p. 82)

$f(x) = \text{int}(x)$



Things to Know

Function (pp. 43–46)

A relation between two sets such that each element x in the first set, the domain, has corresponding to it exactly one element y in the second set. The range is the set of y-values of the function for the x-values in the domain.

A function can also be characterized as a set of ordered pairs (x, y) in which no first element is paired with two different second elements.

Function notation (pp. 46–49)

$y = f(x)$

f is a symbol for the function.

x is the argument, or independent variable.

y is the dependent variable.

$f(x)$ is the value of the function at x, or the image of x.

A function f may be defined implicitly by an equation involving x and y or explicitly by writing $y = f(x)$.

Difference quotient of f (pp. 48 and 77)

$\frac{f(x+h) - f(x)}{h} \quad h \neq 0$

Domain (pp. 49–51)

If unspecified, the domain of a function f defined by an equation is the largest set of real numbers for which $f(x)$ is a real number.

Vertical-line test (p. 57)

A set of points in the plane is the graph of a function if and only if every vertical line intersects the graph in at most one point.

Even function f (p. 66)

$f(-x) = f(x)$ for every x in the domain ($-x$ must also be in the domain).

Odd function f (p. 66)

$f(-x) = -f(x)$ for every x in the domain ($-x$ must also be in the domain).

Increasing function (p. 68)

A function f is increasing on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

Decreasing function (p. 68)

A function f is decreasing on an open interval I if, for any choice of x_1 and x_2 in I, with $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

Constant function (p. 68)

A function f is constant on an open interval I if, for all choices of x in I, the values of $f(x)$ are equal.

Local maximum (p. 69)

A function f has a local maximum at c if there is an open interval I containing c so that, for all x in I, $f(x) \leq f(c)$.

Local minimum (p. 69)

A function f has a local minimum at c if there is an open interval I containing c so that, for all x in I, $f(x) \geq f(c)$.

Absolute maximum and Absolute minimum (p. 70)

Let f denote a function defined on some interval I.

If there is a number u in I for which $f(x) \leq f(u)$ for all x in I, then $f(u)$ is the absolute maximum of f on I and we say the absolute maximum of f occurs at u.

If there is a number v in I for which $f(x) \geq f(v)$, for all x in I, then $f(v)$ is the absolute minimum of f on I and we say the absolute minimum of f occurs at v.

Average rate of change of a function (p. 72)

The average rate of change of f from a to b is

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} \quad a \neq b$$

Direct variation (p. 106)

$y = kx, k \neq 0$

Inverse variation (p. 107)

$y = \frac{k}{x}, k \neq 0$

Objectives

Section		You should be able to...	Examples	Review Exercises
1.1	1	Determine whether a relation represents a function (p. 43)	1–5	1, 2
	2	Find the value of a function (p. 46)	6, 7	3–5, 15, 39
	3	Find the domain of a function defined by an equation (p. 49)	8, 9	6–11
	4	Form the sum, difference, product, and quotient of two functions (p. 51)	10	12–14
1.2	1	Identify the graph of a function (p. 57)	1	27, 28
	2	Obtain information from or about the graph of a function (p. 58)	2–5	16(a)–(e), 17(a), 17(e), 17(g)
1.3	1	Determine even and odd functions from a graph (p. 66)	1	17(f)
	2	Identify even and odd functions from the equation (p. 67)	2	18–21
	3	Use a graph to determine where a function is increasing, decreasing, or constant (p. 68)	3	17(b)
	4	Use a graph to locate local maxima and local minima (p. 69)	4	17(c)
	5	Use a graph to locate the absolute maximum and the absolute minimum (p. 70)	5	17(d)
	6	Use a graphing utility to approximate local maxima and local minima and to determine where a function is increasing or decreasing (p. 71)	6	22, 23, 40(d), 41(b)
	7	Find the average rate of change of a function (p. 72)	7, 8	24–26
1.4	1	Graph the functions listed in the library of functions (p. 78)	1, 2	29, 30
	2	Graph piecewise-defined functions (p. 83)	3, 4	37, 38
1.5	1	Graph functions using vertical and horizontal shifts (p. 89)	1–5, 11, 12, 13	16(f), 31, 33, 34, 35, 36
	2	Graph functions using compressions and stretches (p. 92)	6–8, 12	16(g), 32, 36
	3	Graph functions using reflections about the x-axis or y-axis (p. 94)	9–11, 13	16(h), 32, 34, 36
1.6	1	Build and analyze functions (p. 101)	1–3	40, 41
1.7	1	Construct a model using direct variation (p. 106)	1	42
	2	Construct a model using inverse variation (p. 107)	2	43
	3	Construct a model using joint or combined variation (p. 107)	3,4	44

Review Exercises

In Problems 1 and 2, determine whether each relation represents a function. For each function, state the domain and range.

1. $\{(-1,0),(2,3),(4,0)\}$ **2.** $\{(4,-1),(2,1),(4,2)\}$

In Problems 3–5, find the following for each function:

(a) $f(2)$ (b) $f(-2)$ (c) $f(-x)$ (d) $-f(x)$ (e) $f(x-2)$ (f) $f(2x)$

3. $f(x)=\dfrac{3x}{x^2-1}$ **4.** $f(x)=\sqrt{x^2-4}$ **5.** $f(x)=\dfrac{x^2-4}{x^2}$

In Problems 6–11, find the domain of each function.

6. $f(x)=\dfrac{x}{x^2-9}$ **7.** $f(x)=\sqrt{2-x}$ **8.** $g(x)=\dfrac{|x|}{x}$

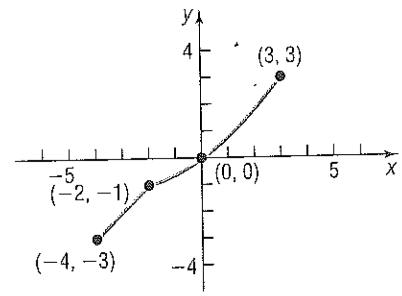
9. $f(x)=\dfrac{x}{x^2+2x-3}$ **10.** $f(x)=\dfrac{\sqrt{x+1}}{x^2-4}$ **11.** $g(x)=\dfrac{x}{\sqrt{x+8}}$

In Problems 12–14, find $f+g$, $f-g$, $f\cdot g$, and $\dfrac{f}{g}$ for each pair of functions. State the domain of each of these functions.

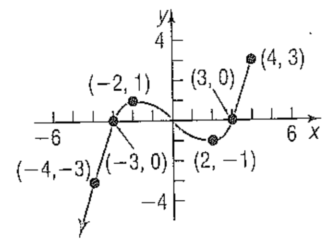
12. $f(x)=2-x;\quad g(x)=3x+1$ **13.** $f(x)=3x^2+x+1;\quad g(x)=3x$ **14.** $f(x)=\dfrac{x+1}{x-1};\quad g(x)=\dfrac{1}{x}$

15. Find the difference quotient of $f(x)=-2x^2+x+1$; that is, find $\dfrac{f(x+h)-f(x)}{h}$, $h\neq 0$.

16. Use the graph of the function f shown to:
   (a) Find the domain and the range of f.
   (b) List the intercepts.
   (c) Find $f(-2)$.
   (d) Find the value(s) of x for which $f(x) = -3$.
   (e) Solve $f(x) > 0$.
   (f) Graph $y = f(x - 3)$.
   (g) Graph $y = f\left(\frac{1}{2}x\right)$.
   (h) Graph $y = -f(x)$.



17. Use the graph of the function f shown to find:
   (a) The domain and the range of f.
   (b) The intervals on which f is increasing, decreasing, or constant.
   (c) The local minimum values and local maximum values.
   (d) The absolute maximum and absolute minimum.
   (e) Whether the graph is symmetric with respect to the x-axis, the y-axis, or the origin.
   (f) Whether the function is even, odd, or neither.
   (g) The intercepts, if any.



In Problems 18–21, determine (algebraically) whether the given function is even, odd, or neither.

18. $f(x) = x^3 - 4x$ **19.** $g(x) = \dfrac{4 + x^2}{1 + x^4}$ **20.** $G(x) = 1 - x + x^3$ **21.** $f(x) = \dfrac{x}{1 + x^2}$

In Problems 22 and 23, use a graphing utility to graph each function over the indicated interval. Approximate any local maximum values and local minimum values. Determine where the function is increasing and where it is decreasing.

22. $f(x) = 2x^3 - 5x + 1 \quad (-3, 3)$ **23.** $f(x) = 2x^4 - 5x^3 + 2x + 1 \quad (-2, 3)$

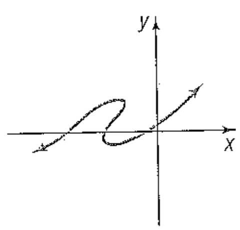
24. Find the average rate of change of $f(x) = 8x^2 - x$.
   (a) From 1 to 2 (b) From 0 to 1 (c) From 2 to 4

In Problems 25 and 26, find the average rate of change from 2 to 3 for each function f. Be sure to simplify.

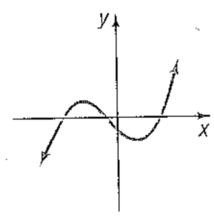
25. $f(x) = 2 - 5x$ **26.** $f(x) = 3x - 4x^2$

In Problems 27 and 28, is the graph shown the graph of a function?

27.



28.



In Problems 29 and 30, sketch the graph of each function. Be sure to label at least three points.

29. $f(x) = |x|$ **30.** $f(x) = \sqrt{x}$

In Problems 31–36, graph each function using the techniques of shifting, compressing or stretching, and reflections. Identify any intercepts on the graph. State the domain and, based on the graph, find the range.

31. $F(x) = |x| - 4$ **32.** $g(x) = -2|x|$ **33.** $h(x) = \sqrt{x - 1}$
34. $f(x) = \sqrt{1 - x}$ **35.** $h(x) = (x - 1)^2 + 2$ **36.** $g(x) = -2(x + 2)^3 - 8$

In Problems 37 and 38,
   (a) Find the domain of each function. *(b) Locate any intercepts.* *(c) Graph each function.*
   (d) Based on the graph, find the range. *(e) Is f continuous on its domain?*

37. $f(x) = \begin{cases} 3x & \text{if } -2 < x \le 1 \\ x + 1 & \text{if } x > 1 \end{cases}$

38. $f(x) = \begin{cases} x & \text{if } -4 \le x < 0 \\ 1 & \text{if } x = 0 \\ 3x & \text{if } x > 0 \end{cases}$

39. A function f is defined by

$$f(x) = \frac{Ax + 5}{6x - 2}$$

If $f(1) = 4$, find A.

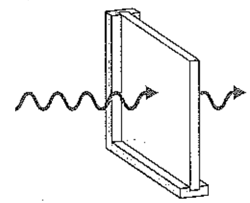
40. Constructing a Closed Box A closed box with a square base is required to have a volume of 10 cubic feet.
(a) Build a model that expresses the amount A of material used to make such a box as a function of the length x of a side of the square base.
(b) How much material is required for a base 1 foot by 1 foot?
(c) How much material is required for a base 2 feet by 2 feet?
(d) Graph $A = A(x)$. For what value of x is A smallest?

41. Area of a Rectangle A rectangle has one vertex in quadrant I on the graph of $y = 10 - x^2$, another at the origin, one on the positive x-axis, and one on the positive y-axis.
(a) Express the area A of the rectangle as a function of x.
(b) Find the largest area A that can be enclosed by the rectangle.

42. Mortgage Payments The monthly payment p on a mortgage varies directly with the amount B borrowed. If the monthly payment on a 30-year mortgage is \$854.00 when \$130,000 is borrowed, find a model that relates the monthly payment p to the amount B borrowed for a mortgage with the same terms. Then find the monthly payment p when the amount borrowed is \$165,000

43. Weight of a Body The weight of a body varies inversely with the square of its distance from the center of Earth. Assuming that the radius of Earth is 3960 miles, how much would a man weigh at an altitude of 1 mile above Earth's surface if he weighs 200 pounds on Earth's surface?

44. Heat Loss The amount of heat transferred per hour through a glass window varies jointly with the surface area of the window and the difference in temperature between the areas separated by the glass. A window with a surface area of 7.5 square feet loses 135 BTU per hour when the temperature difference is 40°F. How much heat is lost per hour for a similar window with a surface area of 12 square feet when the temperature difference is 35°F?

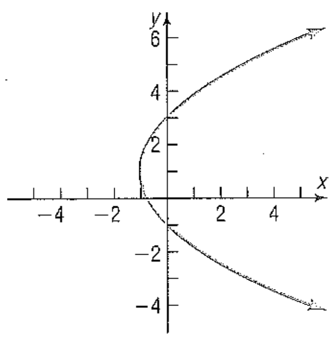


Chapter Test

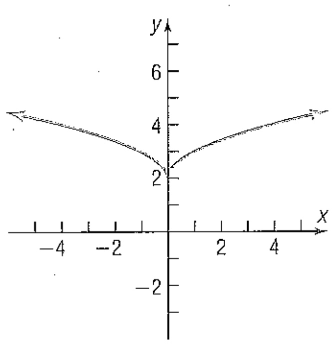
CHAPTER Test Prep VIDEOS

Chapter Test Prep Videos include step-by-step solutions to all chapter test exercises and can be found in MyMathLab or on this text's YouTube Channel. (search "SullivanPrecalcUC3e")

1. Determine whether each relation represents a function. For each function, state the domain and the range.
(a) $\{(2,5), (4,6), (6,7), (8,8)\}$
(b) $\{(1,3), (4,-2), (-3,5), (1,7)\}$
(c)



(d)



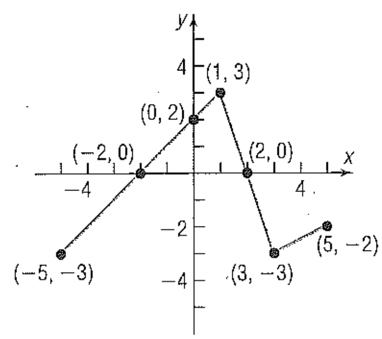
In Problems 2–4, find the domain of each function and evaluate each function at $x = -1$.

2. $f(x) = \sqrt{4 - 5x}$

3. $g(x) = \dfrac{x + 2}{|x + 2|}$

4. $h(x) = \dfrac{x - 4}{x^2 + 5x - 36}$

5. Using the graph of the function f:



(a) Find the domain and the range of f.
(b) List the intercepts.
(c) Find $f(1)$.
(d) For what value(s) of x does $f(x) = -3$?
(e) Solve $f(x) < 0$.

6. Use a graphing utility to graph the function $f(x) = -x^4 + 2x^3 + 4x^2 - 2$ on the interval $(-5, 5)$. Approximate any local maximum values and local minimum values rounded to two decimal places. Determine where the function is increasing and where it is decreasing.

7. Consider the function $g(x) = \begin{cases} 2x + 1 & \text{if } x < -1 \\ x - 4 & \text{if } x \geq -1 \end{cases}$
(a) Graph the function.
(b) List the intercepts.
(c) Find $g(-5)$.
(d) Find $g(2)$.

8. For the function $f(x) = 3x^2 - 2x + 4$, find the average rate of change of f from 3 to 4.

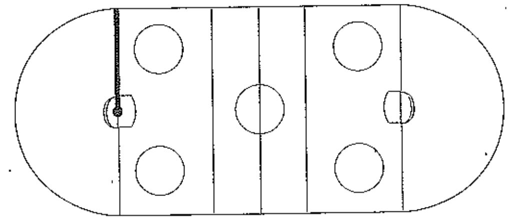
9. For the functions $f(x) = 2x^2 + 1$ and $g(x) = 3x - 2$, find the following and simplify:
   (a) $f - g$
   (b) $f \cdot g$
   (c) $f(x + h) - f(x)$

10. Graph each function using the techniques of shifting, compressing or stretching, and reflections. Start with the graph of the basic function and show all stages.
   (a) $h(x) = -2(x + 1)^3 + 3$
   (b) $g(x) = |x + 4| + 2$

11. The variable interest rate on a student loan changes each July 1 based on the bank prime loan rate. For the years 1992–2007, this rate can be approximated by the model $r(x) = -0.115x^2 + 1.183x + 5.623$, where x is the number of years since 1992 and r is the interest rate as a percent.
   (a) Use a graphing utility to estimate the highest rate during this time period. During which year was the interest rate the highest?
   (b) Use the model to estimate the rate in 2010. Does this value seem reasonable?

   Source: U.S. Federal Reserve

12. A community skating rink is in the shape of a rectangle with semicircles attached at the ends. The length of the rectangle is 20 feet less than twice the width. The thickness of the ice is 0.75 inch.
   (a) Build a model that expresses the ice volume, V, as a function of the width, x.
   (b) How much ice is in the rink if the width is 90 feet?



13. The resistance (in ohms) of a circular conductor varies directly with the length of the conductor and inversely with the square of the radius of the conductor. If 50 feet of wire with a radius of 6×10^{-3} inch has a resistance of 10 ohms, what would be the resistance of 100 feet of the same wire if the radius were increased to 7×10^{-3} inch?

Chapter Projects

Internet-based Project

I. Choosing a Cellular Telephone Plan Collect information from your family, friends, or consumer agencies such as Consumer Reports. Then decide on a cellular telephone provider, choosing the company that you feel offers the best service. Once you have selected a service provider, research the various types of individual plans offered by the company by visiting the provider's website.

1. Suppose you expect to use 400 anytime minutes without a texting or data plan. What would be the monthly cost of each plan you are considering?
2. Suppose you expect to use 600 anytime minutes with unlimited texting, but no data plan. What would be the monthly cost of each plan you are considering?
3. Suppose you expect to use 500 anytime minutes with unlimited texting and an unlimited data plan. What would be the monthly cost of each plan you are considering?
4. Suppose you expect to use 500 anytime minutes with unlimited texting and 20 MB of data. What would be the monthly cost of each plan you are considering?
5. Build a model that describes the monthly cost C as a function of the number of anytime minutes used m, assuming unlimited texting and 20 MB of data each month for each plan you are considering.
6. Graph each function from Problem 5.
7. Based on your particular usage, which plan is best for you?
8. Now, develop an Excel spreadsheet to analyze the various plans you are considering. Suppose you want a plan that offers 700 anytime minutes with additional minutes costing \$0.40 per minute that costs \$39.99 per month. In addition, you want unlimited texting, which costs an additional \$20 per month, and a data plan that offers up to 25 MB of data each month, with each additional MB costing \$0.20. Because cellular telephone plans' cost structure is based on piecewise-defined functions, we need "if-then" statements within Excel to analyze the cost of the plan. Use the Excel spreadsheet on the next page as a guide in developing your worksheet. Enter into your spreadsheet a variety of possible minutes and data used to help arrive at a decision regarding which plan is best for you.
9. Write a paragraph supporting the choice in plans that best meets your needs.
10. How are "if/then" loops similar to a piecewise-defined function?

Citation: Excel © 2010 Microsoft Corporation. Used with permission from Microsoft.

	A	B	C	D
1				
2	Monthly fee	$ 39.99		
3	Allotted number of anytime minutes	700		
4	Number of anytime minutes used	700		
5	Cost per additional minute	$ 0.40		
6	Monthly cost of text messaging	$ 20.00		
7	Monthly cost of data plan	$ 9.99		
8	Allotted data per month (MB)	25		
9	Data used	30		
10	Cost per additional MB of data	$ 0.20		
11				
12	Cost of phone minutes	=IF(B4<B3,B2,B2+B5*(B4-B3))		
13	Cost of data	=IF(B9<B8,B7,B7+B10*(B9-B8))		
14				
15	Total Cost	=B6+B12+B13		
16				

The following projects are available on the Instructor's Resource Center (IRC).

II. **Project at Motorola:** ***Wireless Internet Service*** Use functions and their graphs to analyze the total cost of various wireless Internet service plans.

III. **Cost of Cable** When government regulations and customer preference influence the path of a new cable line, the Pythagorean Theorem can be used to assess the cost of installation.

IV. **Oil Spill** Functions are used to analyze the size and spread of an oil spill from a leaking tanker.