EXAMPLE 12 A Cubic Function of Best Fit

The data in Table 6 represent the weekly cost C (in thousands of dollars) of printing x thousand textbooks.

(a) Draw a scatter diagram of the data using x as the independent variable and C as the dependent variable. Comment on the type of relation that may exist between the variables x and C.

(b) Using a graphing utility, find the cubic function of best fit $C = C(x)$ that models the relation between number of texts and cost.

(c) Graph the cubic function of best fit on your scatter diagram.

(d) Use the function found in part (b) to predict the cost of printing 22 thousand texts per week.

Table 6

Number of Textbooks, x (thousands)	Cost, C (\$1000s)
0	100
5	128.1
10	144
13	153.5
17	161.2
18	162.6
20	166.3
23	178.9
25	190.2
27	221.8

Solution

(a) Figure 22 shows the scatter diagram. A cubic relation may exist between the two variables.

(b) Upon executing the CUBIC REGression program, we obtain the results shown in Figure 23. The output that the utility provides shows us the equation $y = ax^3 + bx^2 + cx + d$. The cubic function of best fit to the data is $C(x) = 0.0155x^3 - 0.5951x^2 + 9.1502x + 98.4327$.

(c) Figure 24 shows the graph of the cubic function of best fit on the scatter diagram. The function fits the data reasonably well.

Figure 22

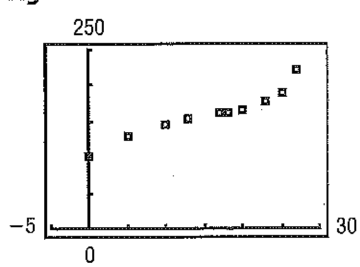


Figure 23

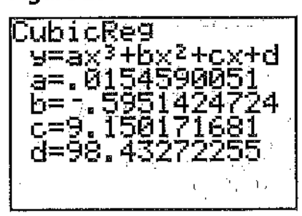
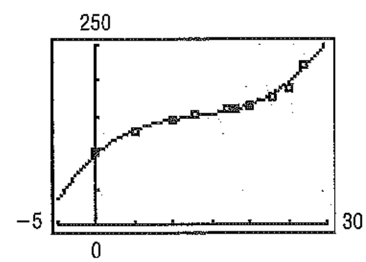


Figure 24



(d) We evaluate the function $C(x)$ at $x = 22$.

$$C(22) = 0.0155(22)^3 - 0.5951(22)^2 + 9.1502(22) + 98.4327 \approx 176.8$$

The model predicts that the cost of printing 22 thousand textbooks in a week will be 176.8 thousand dollars—that is \$176,800.

3.1 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The intercepts of the equation $9x^2 + 4y = 36$ are ________. (pp. 11–12)
2. Is the expression $4x^3 - 3.6x^2 - \sqrt{2}$ a polynomial? If so, what is its degree? (pp. A22–A24)
3. To graph $y = x^2 - 4$, you would shift the graph of $y = x^2$ ________ a distance of ________ units. (pp. 89–90)
4. Use a graphing utility to approximate (rounded to two decimal places) the local maxima and the local minima of $f(x) = x^3 - 2x^2 - 4x + 5$, for $-3 < x < 3$. (pp. 71–72)
5. ***True or False*** The y-intercepts of the graph of a function are also the zeros of the function. (pp. 59–60)
6. If $g(5) = 0$, what point is on the graph of g? What is the x-intercept of the graph of g? (pp. 59–60)

Concepts and Vocabulary

7. The graph of every polynomial function is both __________ and __________.

8. If r is a real zero of even multiplicity of a function f, then the graph of f __________ (crosses/touches) the x-axis at r.

9. The graphs of power functions of the form $f(x) = x^n$, where n is an even integer, always contain the points __________, __________, and __________.

10. If r is a solution to the equation $f(x) = 0$, name three additional statements that can be made about f and r, assuming f is a polynomial function.

11. The points at which a graph changes direction (from increasing to decreasing or decreasing to increasing) are called __________.

12. The graph of the function $f(x) = 3x^4 - x^3 + 5x^2 - 2x - 7$ will behave like the graph of __________ for large values of $|x|$.

13. If $f(x) = -2x^5 + x^3 - 5x^2 + 7$, then $\lim_{x\to-\infty} f(x) =$ __________ and $\lim_{x\to\infty} f(x) =$ __________.

14. Explain what the notation $\lim_{x\to\infty} f(x) = -\infty$ means.

Skill Building

In Problems 15–26, determine which functions are polynomial functions. For those that are, state the degree. For those that are not, tell why not. Write each polynomial in standard form. Then identify the leading term and the constant term.

15. $f(x) = 4x + x^3$
16. $f(x) = 5x^2 + 4x^4$
17. $g(x) = \dfrac{1 - x^2}{2}$
18. $h(x) = 3 - \dfrac{1}{2}x$
19. $f(x) = 1 - \dfrac{1}{x}$
20. $f(x) = x(x - 1)$
21. $g(x) = x^{3/2} - x^2 + 2$
22. $h(x) = \sqrt{x}(\sqrt{x} - 1)$
23. $f(x) = 5x^4 - \pi x^3 + \dfrac{1}{2}$
24. $f(x) = \dfrac{x^2 - 5}{x^3}$
25. $G(x) = 2(x - 1)^2(x^2 + 1)$
26. $G(x) = -3x^2(x + 2)^3$

In Problems 27–40, use transformations of the graph of $y = x^4$ or $y = x^5$ to graph each function.

27. $f(x) = (x + 1)^4$
28. $f(x) = (x - 2)^5$
29. $f(x) = x^5 - 3$
30. $f(x) = x^4 + 2$
31. $f(x) = \dfrac{1}{2}x^4$
32. $f(x) = 3x^5$
33. $f(x) = -x^5$
34. $f(x) = -x^4$
35. $f(x) = (x - 1)^5 + 2$
36. $f(x) = (x + 2)^4 - 3$
37. $f(x) = 2(x + 1)^4 + 1$
38. $f(x) = \dfrac{1}{2}(x - 1)^5 - 2$
39. $f(x) = 4 - (x - 2)^5$
40. $f(x) = 3 - (x + 2)^4$

In Problems 41–48, form a polynomial function whose real zeros and degree are given. Answers will vary depending on the choice of a leading coefficient.

41. Zeros: $-1, 1, 3$; degree 3
42. Zeros: $-2, 2, 3$; degree 3
43. Zeros: $-3, 0, 4$; degree 3
44. Zeros: $-4, 0, 2$; degree 3
45. Zeros: $-4, -1, 2, 3$; degree 4
46. Zeros: $-3, -1, 2, 5$; degree 4
47. Zeros: -1, multiplicity 1; 3, multiplicity 2; degree 3
48. Zeros: -2, multiplicity 2; 4, multiplicity 1; degree 3

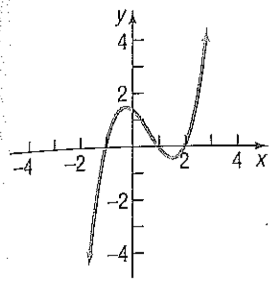
In Problems 49–60, for each polynomial function:

(a) List each real zero and its multiplicity.
(b) Determine whether the graph crosses or touches the x-axis at each x-intercept.
(c) Determine the maximum number of turning points on the graph.
(d) Determine the end behavior; that is, find the power function that the graph of f resembles for large values of $|x|$.

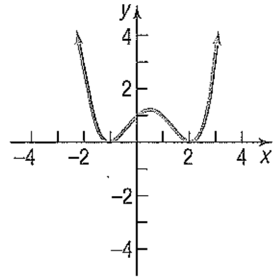
49. $f(x) = 3(x - 7)(x + 3)^2$
50. $f(x) = 4(x + 4)(x + 3)^3$
51. $f(x) = 4(x^2 + 1)(x - 2)^3$
52. $f(x) = 2(x - 3)(x^2 + 4)^3$
53. $f(x) = -2\left(x + \dfrac{1}{2}\right)^2(x + 4)^3$
54. $f(x) = \left(x - \dfrac{1}{3}\right)^2(x - 1)^3$
55. $f(x) = (x - 5)^3(x + 4)^2$
56. $f(x) = (x + \sqrt{3})^2(x - 2)^4$
57. $f(x) = 3(x^2 + 8)(x^2 + 9)^2$
58. $f(x) = -2(x^2 + 3)^3$
59. $f(x) = -2x^2(x^2 - 2)$
60. $f(x) = 4x(x^2 - 3)$

In Problems 61–64, identify which of the graphs could be the graph of a polynomial function. For those that could, list the real zeros and state the least degree the polynomial can have. For those that could not, say why not.

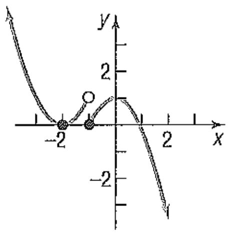
61.



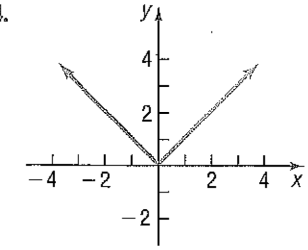
62.



63.

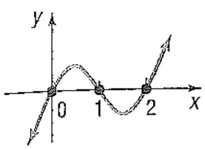


64.

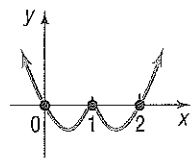


In Problems 65–68, construct a polynomial function that might have the given graph. (More than one answer may be possible.)

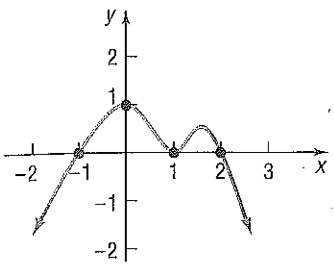
65.



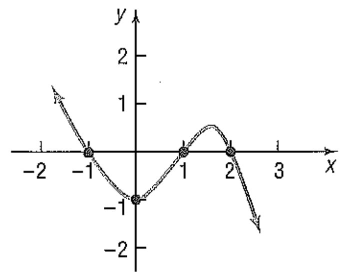
66.



67.

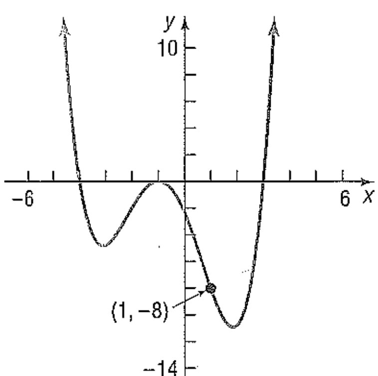


68.

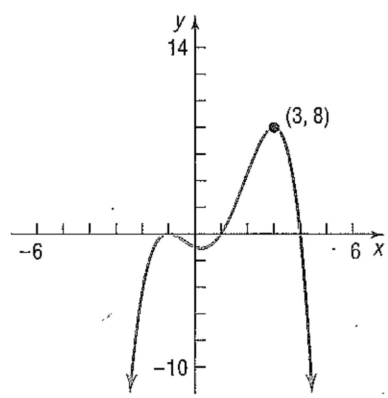


In Problems 69–72, write a polynomial function whose graph is shown (use the smallest degree possible).

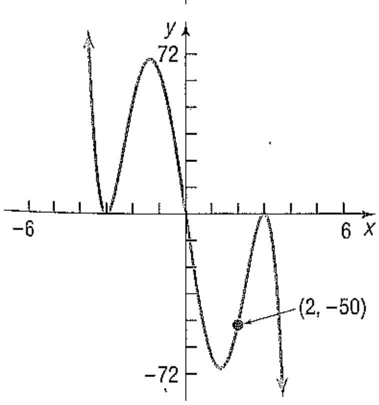
69.



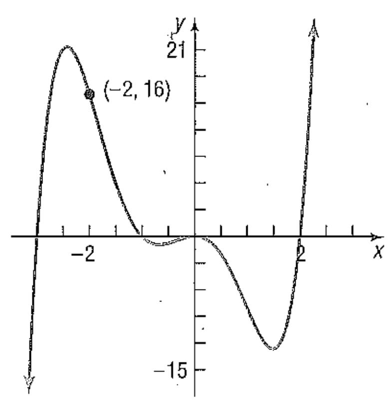
70.



71.



72.



In Problems 73–96, analyze each polynomial function by following Steps 1 through 5 on page 205.

73. $f(x) = x^2(x - 3)$ **74.** $f(x) = x(x + 2)^2$ **75.** $f(x) = (x + 4)(x - 2)^2$

76. $f(x) = (x - 1)(x + 3)^2$ **77.** $f(x) = -2(x + 2)(x - 2)^3$ **78.** $f(x) = -\frac{1}{2}(x + 4)(x - 1)^3$

79. $f(x) = (x + 4)^2(1 - x)$ **80.** $f(x) = (3 - x)(2 + x)(x + 1)$ **81.** $f(x) = (x + 1)(x - 2)(x + 4)$

82. $f(x) = (x - 1)(x + 4)(x - 3)$
83. $f(x) = x^2(x - 2)(x + 2)$
84. $f(x) = x^2(x - 3)(x + 4)$
85. $f(x) = (x + 1)^2(x - 2)^2$
86. $f(x) = (x + 2)^2(x - 4)^2$
87. $f(x) = x(1 - x)(2 - x)$
88. $f(x) = x(3 - x)^2$
89. $f(x) = x^2(x + 3)(x + 1)$
90. $f(x) = x^2(x - 3)(x - 1)$
91. $f(x) = (x - 2)^2(x + 2)(x + 4)$
92. $f(x) = (x + 1)^3(x - 3)$
93. $f(x) = 5x(x^2 - 4)(x + 3)$
94. $f(x) = -2(x - 1)^2(x^2 - 16)$
95. $f(x) = x^2(x - 2)(x^2 + 3)$
96. $f(x) = x^2(x^2 + 1)(x + 4)$

In Problems 97–104, analyze each polynomial function f by following Steps 1 through 8 on page 206.

97. $f(x) = x^3 + 0.2x^2 - 1.5876x - 0.31752$
98. $f(x) = x^3 - 0.8x^2 - 4.6656x + 3.73248$
99. $f(x) = x^3 + 2.56x^2 - 3.31x + 0.89$
100. $f(x) = x^3 - 2.91x^2 - 7.668x - 3.8151$
101. $f(x) = x^4 - 2.5x^2 + 0.5625$
102. $f(x) = x^4 - 18.5x^2 + 50.2619$
103. $f(x) = 2x^4 - \pi x^3 + \sqrt{5}x - 4$
104. $f(x) = -1.2x^4 + 0.5x^2 - \sqrt{3}x + 2$

Mixed Practice

In Problems 105–112, analyze each polynomial function by following Steps 1 through 5 on page 205.
[**Hint:** You will need to first factor the polynomial].

105. $f(x) = 4x - x^3$
106. $f(x) = x - x^3$
107. $f(x) = x^3 + x^2 - 12x$
108. $f(x) = x^3 + 2x^2 - 8x$
109. $f(x) = 2x^4 + 12x^3 - 8x^2 - 48x$
110. $f(x) = 4x^3 + 10x^2 - 4x - 10$
111. $f(x) = -x^5 - x^4 + x^3 + x^2$
112. $f(x) = -x^5 + 5x^4 + 4x^3 - 20x^2$

In Problems 113–116, construct a polynomial function f with the given characteristics.

113. Zeros: $-3, 1, 4$; degree 3; y-intercept: 36
114. Zeros: $-4, -1, 2$; degree 3; y-intercept: 16
115. Zeros: -5 (multiplicity 2); 2(multiplicity 1); 4 (multiplicity 1); degree 4; contains the point $(3, 128)$
116. Zeros: -4 (multiplicity 1); 0 (multiplicity 3); 2 (multiplicity 1); degree 5; contains the point $(-2, 64)$

117. $G(x) = (x + 3)^2(x - 2)$
(a) Identify the x-intercepts of the graph of G.
(b) What are the x-intercepts of the graph of $y = G(x + 3)$?

118. $h(x) = (x + 2)(x - 4)^3$
(a) Identify the x-intercepts of the graph of h.
(b) What are the x-intercepts of the graph of $y = h(x - 2)$?

Applications and Extensions

119. Hurricanes In 2012, Hurricane Sandy struck the East Coast of the United States, killing 147 people and causing an estimated $75 billion in damage. With a gale diameter of about 1000 miles, it was the largest ever to form over the Atlantic Basin. The following data represent the number of major hurricane strikes in the Atlantic Basin (category 3, 4, or 5) each decade from 1921 to 2010.

Decade, *x*	Major Hurricanes Striking Atlantic Basin, *H*
1921–1930, 1	17
1931–1940, 2	16
1941–1950, 3	29
1951–1960, 4	33
1961–1970, 5	27
1971–1980, 6	16
1981–1990, 7	16
1991–2000, 8	27
2001–2010, 9	33

Source: National Oceanic & Atmospheric Administration

(a) Draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.
(b) Use a graphing utility to find the cubic function of best fit that models the relation between decade and number of major hurricanes.
(c) Use the model found in part (b) to predict the number of major hurricanes that struck the Atlantic Basin between 1961 and 1970.
(d) With a graphing utility, draw a scatter diagram of the data and then graph the cubic function of best fit on the scatter diagram.
(e) Concern has risen about the increase in the number and intensity of hurricanes, but some scientists believe this is just a natural fluctuation that could last another decade or two. Use your model to predict the number of major hurricanes that will strike the Atlantic Basin between 2011 and 2020. Is your result reasonable?

120. **Cost of Manufacturing** The following data represent the cost C (in thousands of dollars) of manufacturing Chevy Cobalts and the number x of Cobalts produced.

Number of Cobalts Produced, x	Cost, C
0	10
1	23
2	31
3	38
4	43
5	50
6	59
7	70
8	85
9	105
10	135

(a) Draw a scatter diagram of the data using x as the independent variable and C as the dependent variable. Comment on the type of relation that may exist between the variables C and x.
(b) Use a graphing utility to find the cubic function of best fit $C = C(x)$.
(c) Graph the cubic function of best fit on the scatter diagram.
(d) Use the function found in part (b) to predict the cost of manufacturing 11 Cobalts.
(e) Interpret the y-intercept.

121. **Temperature** The data in the next column represent the temperature T (Fahrenheit) in Kansas City, Missouri, x hours after midnight on March 28, 2013.

(a) Draw a scatter diagram of the data. Comment on the type of relation that may exist between the two variables.
(b) Find the average rate of change in temperature from 9 AM to 12 noon.
(c) What is the average rate of change in temperature from 6 PM to 9 PM?
(d) Decide on a function of best fit to these data (linear, quadratic, or cubic) and use this function to predict the temperature at 5 PM.
(e) With a graphing utility, draw a scatter diagram of the data and then graph the function of best fit on the scatter diagram.
(f) Interpret the y-intercept.

Hours after Midnight, x	Temperature (F), T
3	40
6	39
9	41
12	55
15	60
18	64
21	57
24	42

Source: The Weather Underground

122. **A Geometric Series** In calculus, you will learn that certain functions can be approximated by polynomial functions. We will explore one such function now.

(a) Using a graphing utility, create a table of values with $Y_1 = f(x) = \dfrac{1}{1-x}$ and $Y_2 = g_2(x) = 1 + x + x^2 + x^3$ for $-1 < x < 1$ with $\Delta\text{Tbl} = 0.1$.
(b) Using a graphing utility, create a table of values with $Y_1 = f(x) = \dfrac{1}{1-x}$ and $Y_2 = g_3(x) = 1 + x + x^2 + x^3 + x^4$ for $-1 < x < 1$ with $\Delta\text{Tbl} = 0.1$.
(c) Using a graphing utility, create a table of values with $Y_1 = f(x) = \dfrac{1}{1-x}$ and $Y_2 = g_4(x) = 1 + x + x^2 + x^3 + x^4 + x^5$ for $-1 < x < 1$ with $\Delta\text{Tbl} = 0.1$.
(d) What do you notice about the values of the function as more terms are added to the polynomial? Are there some values of x for which the approximations are better?

123. **Future Value of Money** Suppose that you make a deposit of \$500 at the beginning of every year into an Individual Retirement Account (IRA) earning interest r. At the beginning of the first year, the value of the account will be \$500; at the beginning of the second year, the value of the account, will be

$$\underbrace{\$500 + \$500r}_{\text{Value of 1st deposit}} + \underbrace{\$500}_{\text{Value of 2nd deposit}} = \$500(1 + r) + \$500 = \$500r + \$1000$$

(a) Verify that the value of the account at the beginning of the third year is $T(r) = 500r^2 + 1500r + 1500$.
(b) The account value at the beginning of the fourth year is $F(r) = 500r^3 + 2000r^2 + 3000r + 2000$. If the annual rate of interest is $5\% = 0.05$, what will be the value of the account at the beginning of the fourth year?

Discussion and Writing

124. Can the graph of a polynomial function have no y-intercept? Can it have no x-intercepts? Explain.

125. Write a few paragraphs that provide a general strategy for graphing a polynomial function. Be sure to mention the following: degree, intercepts, end behavior, and turning points.

126. Make up a polynomial that has the following characteristics: crosses the x-axis at -1 and 4, touches the x-axis at 0 and 2, and is above the x-axis between 0 and 2. Give your polynomial to a fellow classmate and ask for a written critique.

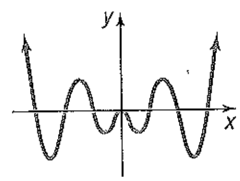
127. Make up two polynomials, not of the same degree, with the following characteristics: crosses the x-axis at -2, touches the x-axis at 1, and is above the x-axis between -2 and 1. Give your polynomials to a fellow classmate and ask for a written critique.

128. The graph of a polynomial function is always smooth and continuous. Name a function studied earlier that is smooth but not continuous. Name one that is continuous, but not smooth.

129. Which of the following statements are true regarding the graph of the cubic polynomial $f(x) = x^3 + bx^2 + cx + d$? (Give reasons for your conclusions.)

(a) It intersects the y-axis in one and only one point.
(b) It intersects the x-axis in at most three points.
(c) It intersects the x-axis at least once.
(d) For $|x|$ very large, it behaves like the graph of $y = x^3$.
(e) It is symmetric with respect to the origin.
(f) It passes through the origin.

130. The illustration shows the graph of a polynomial function.



(a) Is the degree of the polynomial even or odd?
(b) Is the leading coefficient positive or negative?
(c) Is the function even, odd, or neither?
(d) Why is x^2 necessarily a factor of the polynomial?
(e) What is the minimum degree of the polynomial?
(f) Formulate five different polynomials whose graphs could look like the one shown. Compare yours to those of other students. What similarities do you see? What differences?

131. Design a polynomial function with the following characteristics: degree 6; four distinct real zeros, one of multiplicity 3; y-intercept 3; behaves like $y = -5x^6$ for large values of $|x|$. Is this polynomial unique? Compare your polynomial with those of other students. What terms will be the same as everyone else's? Add some more characteristics, such as symmetry or naming the real zeros. How does this modify the polynomial?

Retain Your Knowledge

Problems 132–135 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

132. Find the equation of the line that contains the point $(2, -3)$ and is perpendicular to the line $5x - 2y = 6$.

133. Find the domain of the function $h(x) = \dfrac{x - 3}{x + 5}$.

134. Use the quadratic formula to find the zeros of the function $f(x) = 4x^2 + 8x - 3$.

135. Solve: $|5x - 3| = 7$

'Are You Prepared?' Answers

1. $(-2,0), (2,0), (0,9)$ **2.** Yes; 3 **3.** Down; 4 **4.** Local maximum 6.48 at $x = -0.67$; local minimum -3 at $x = 2$
5. False **6.** $(5,0)$; 5

3.2 The Real Zeros of a Polynomial Function

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Evaluating Functions (Section 1.1, pp. 46–48)
- Factoring Polynomials (Appendix A, Section A.4, pp. A32–A38)
- Synthetic Division (Appendix A, Section A.5, pp. A41–A44)
- Polynomial Division (Appendix A, Section A.3, pp. A27–A30)
- Zeros of a Quadratic Function (Section 2.3, pp. 138–143)

Now Work the 'Are You Prepared?' problems on page 223.

OBJECTIVES
1 Use the Remainder and Factor Theorems (p. 213)
2 Use Descartes' Rule of Signs to Determine the Number of Positive and the Number of Negative Real Zeros of a Polynomial Function (p. 215)
3 Use the Rational Zeros Theorem to List the Potential Rational Zeros of a Polynomial Function (p. 216)
4 Find the Real Zeros of a Polynomial Function (p. 217)
5 Solve Polynomial Equations (p. 219)
6 Use the Theorem for Bounds on Zeros (p. 220)
7 Use the Intermediate Value Theorem (p. 221)