Historical Feature

Formulas for the solution of third- and fourth-degree polynomial equations exist, and although they are not very practical, they do have an interesting history.

In the 1500s in Italy, mathematical contests were a popular pastime, and people who possessed methods for solving problems kept them secret. (Solutions that were published were already common knowledge.) Niccolo of Brescia (1499–1557), commonly referred to as Tartaglia ("the stammerer"), had the secret for solving cubic (third-degree) equations, which gave him a decided advantage in the contests. Girolamo Cardano (1501–1576) found out that Tartaglia had the secret, and, being interested in cubics, he requested it from Tartaglia. The reluctant Tartaglia hesitated for some time, but finally, swearing Cardano to secrecy with midnight oaths by candlelight, told him the secret. Cardano then published the solution in his book *Ars Magna* (1545), giving Tartaglia the credit but rather compromising the secrecy. Tartaglia exploded into bitter recriminations, and each wrote pamphlets that reflected on the other's mathematics, moral character, and ancestry.

The quartic (fourth-degree) equation was solved by Cardano's student Lodovico Ferrari, and this solution also was included, with credit and this time with permission, in the *Ars Magna*.

Attempts were made to solve the fifth-degree equation in similar ways, all of which failed. In the early 1800s, P. Ruffini, Niels Abel, and Evariste Galois all found ways to show that it is not possible to solve fifth-degree equations by formula, but the proofs required the introduction of new methods. Galois's methods eventually developed into a large part of modern algebra.

Historical Problems

Problems 1–8 develop the Tartaglia–Cardano solution of the cubic equation and show why it is not altogether practical.

1. Show that the general cubic equation $y^3 + by^2 + cy + d = 0$ can be transformed into an equation of the form $x^3 + px + q = 0$ by using the substitution $y = x - \frac{b}{3}$.
2. In the equation $x^3 + px + q = 0$, replace x by $H + K$. Let $3HK = -p$, and show that $H^3 + K^3 = -q$.
3. Based on Problem 2, we have the two equations

$$3HK = -p \quad \text{and} \quad H^3 + K^3 = -q$$

Solve for K in $3HK = -p$ and substitute into $H^3 + K^3 = -q$. Then show that

$$H = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

[**Hint:** Look for an equation that is quadratic in form.]

4. Use the solution for H from Problem 3 and the equation $H^3 + K^3 = -q$ to show that

$$K = \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

5. Use the results from Problems 2 to 4 to show that the solution of $x^3 + px + q = 0$ is

$$x = \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{-q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

6. Use the result of Problem 5 to solve the equation $x^3 - 6x - 9 = 0$.
7. Use a calculator and the result of Problem 5 to solve the equation $x^3 + 3x - 14 = 0$.
8. Use the methods of this section to solve the equation $x^3 + 3x - 14 = 0$.

3.2 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Find $f(-1)$ if $f(x) = 2x^2 - x$. (pp. 46–48)
2. Factor the expression $6x^2 + x - 2$. (pp. A37–A38)
3. Find the quotient and remainder if $3x^4 - 5x^3 + 7x - 4$ is divided by $x - 3$. (pp. A27–A29 or A41–A44)
4. Find the zeros of $f(x) = x^2 + x - 3$. (pp. 141–143)

Concepts and Vocabulary

5. In the process of polynomial division, (Divisor)(Quotient) + ______ = ______.
6. When a polynomial function f is divided by $x - c$, the remainder is ______.
7. If a function f, whose domain is all real numbers, is even and if 4 is a zero of f, then ____ is also a zero.
8. ***True or False*** Every polynomial function of degree 3 with real coefficients has exactly three real zeros.
9. If f is a polynomial function and $x - 4$ is a factor of f, then $f(4) =$ ___.
10. ***True or False*** If f is a polynomial function of degree 4 and if $f(2) = 5$, then

$$\frac{f(x)}{x-2} = p(x) + \frac{5}{x-2}$$

where $p(x)$ is a polynomial of degree 3.

Skill Building

In Problems 11–20, use the Remainder Theorem to find the remainder when f(x) is divided by x − c. Then use the Factor Theorem to determine whether x − c is a factor of f(x).

11. $f(x) = 4x^3 - 3x^2 - 8x + 4$; $x - 2$
12. $f(x) = -4x^3 + 5x^2 + 8$; $x + 3$

13. $f(x) = 3x^4 - 6x^3 - 5x + 10; x - 2$

14. $f(x) = 4x^4 - 15x^2 - 4; x - 2$

15. $f(x) = 3x^6 + 82x^3 + 27; x + 3$

16. $f(x) = 2x^6 - 18x^4 + x^2 - 9; x + 3$

17. $f(x) = 4x^6 - 64x^4 + x^2 - 15; x + 4$

18. $f(x) = x^6 - 16x^4 + x^2 - 16; x + 4$

19. $f(x) = 2x^4 - x^3 + 2x - 1; x - \frac{1}{2}$

20. $f(x) = 3x^4 + x^3 - 3x + 1; x + \frac{1}{3}$

In Problems 21–32, tell the maximum number of real zeros that each polynomial function may have. Then use Descartes' Rule of Signs to determine how many positive and how many negative zeros each polynomial function may have. Do not attempt to find the zeros.

21. $f(x) = -4x^7 + x^3 - x^2 + 2$

22. $f(x) = 5x^4 + 2x^2 - 6x - 5$

23. $f(x) = 2x^6 - 3x^2 - x + 1$

24. $f(x) = -3x^5 + 4x^4 + 2$

25. $f(x) = 3x^3 - 2x^2 + x + 2$

26. $f(x) = -x^3 - x^2 + x + 1$

27. $f(x) = -x^4 + x^2 - 1$

28. $f(x) = x^4 + 5x^3 - 2$

29. $f(x) = x^5 + x^4 + x^2 + x + 1$

30. $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$

31. $f(x) = x^6 - 1$

32. $f(x) = x^6 + 1$

In Problems 33–44, list the potential rational zeros of each polynomial function. Do not attempt to find the zeros.

33. $f(x) = 3x^4 - 3x^3 + x^2 - x + 1$

34. $f(x) = x^5 - x^4 + 2x^2 + 3$

35. $f(x) = x^5 - 6x^2 + 9x - 3$

36. $f(x) = 2x^5 - x^4 - x^2 + 1$

37. $f(x) = -4x^3 - x^2 + x + 2$

38. $f(x) = 6x^4 - x^2 + 2$

39. $f(x) = 6x^4 - x^2 + 9$

40. $f(x) = -4x^3 + x^2 + x + 6$

41. $f(x) = 2x^5 - x^3 + 2x^2 + 12$

42. $f(x) = 3x^5 - x^2 + 2x + 18$

43. $f(x) = 6x^4 + 2x^3 - x^2 + 20$

44. $f(x) = -6x^3 - x^2 + x + 10$

In Problems 45–56, use the Rational Zeros Theorem to find all the real zeros of each polynomial function. Use the zeros to factor f over the real numbers.

45. $f(x) = x^3 + 2x^2 - 5x - 6$

46. $f(x) = x^3 + 8x^2 + 11x - 20$

47. $f(x) = 2x^3 - x^2 + 2x - 1$

48. $f(x) = 2x^3 + x^2 + 2x + 1$

49. $f(x) = 2x^3 - 4x^2 - 10x + 20$

50. $f(x) = 3x^3 + 6x^2 - 15x - 30$

51. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

52. $f(x) = 2x^4 - x^3 - 5x^2 + 2x + 2$

53. $f(x) = x^4 + x^3 - 3x^2 - x + 2$

54. $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$

55. $f(x) = 4x^4 + 5x^3 + 9x^2 + 10x + 2$

56. $f(x) = 3x^4 + 4x^3 + 7x^2 + 8x + 2$

In Problems 57–68, solve each equation in the real number system.

57. $x^4 - x^3 + 2x^2 - 4x - 8 = 0$

58. $2x^3 + 3x^2 + 2x + 3 = 0$

59. $3x^3 + 4x^2 - 7x + 2 = 0$

60. $2x^3 - 3x^2 - 3x - 5 = 0$

61. $3x^3 - x^2 - 15x + 5 = 0$

62. $2x^3 - 11x^2 + 10x + 8 = 0$

63. $x^4 + 4x^3 + 2x^2 - x + 6 = 0$

64. $x^4 - 2x^3 + 10x^2 - 18x + 9 = 0$

65. $x^3 - \frac{2}{3}x^2 + \frac{8}{3}x + 1 = 0$

66. $x^3 + \frac{3}{2}x^2 + 3x - 2 = 0$

67. $2x^4 - 19x^3 + 57x^2 - 64x + 20 = 0$

68. $2x^4 + x^3 - 24x^2 + 20x + 16 = 0$

In Problems 69–78, find bounds on the real zeros of each polynomial function.

69. $f(x) = x^4 - 3x^2 - 4$
70. $f(x) = x^4 - 5x^2 - 36$
71. $f(x) = x^4 + x^3 - x - 1$
72. $f(x) = x^4 - x^3 + x - 1$
73. $f(x) = 3x^4 + 3x^3 - x^2 - 12x - 12$
74. $f(x) = 3x^4 - 3x^3 - 5x^2 + 27x - 36$
75. $f(x) = 4x^5 - x^4 + 2x^3 - 2x^2 + x - 1$
76. $f(x) = 4x^5 + x^4 + x^3 + x^2 - 2x - 2$
77. $f(x) = -x^4 + 3x^3 - 4x^2 - 2x + 9$
78. $f(x) = -4x^5 + 5x^3 + 9x^2 + 3x - 12$

In Problems 79–84, use the Intermediate Value Theorem to show that each polynomial function has a zero in the given interval.

79. $f(x) = 8x^4 - 2x^2 + 5x - 1;\ [0, 1]$
80. $f(x) = x^4 + 8x^3 - x^2 + 2;\ [-1, 0]$
81. $f(x) = 2x^3 + 6x^2 - 8x + 2;\ [-5, -4]$
82. $f(x) = 3x^3 - 10x + 9;\ [-3, -2]$
83. $f(x) = x^5 - x^4 + 7x^3 - 7x^2 - 18x + 18;\ [1.4, 1.5]$
84. $f(x) = x^5 - 3x^4 - 2x^3 + 6x^2 + x + 2;\ [1.7, 1.8]$

In Problems 85–88, each equation has a solution r in the interval indicated. Use the method of Example 10 to approximate this solution correct to two decimal places.

85. $8x^4 - 2x^2 + 5x - 1 = 0;\ 0 \le r \le 1$
86. $x^4 + 8x^3 - x^2 + 2 = 0;\ -1 \le r \le 0$
87. $2x^3 + 6x^2 - 8x + 2 = 0;\ -5 \le r \le -4$
88. $3x^3 - 10x + 9 = 0;\ -3 \le r \le -2$

In Problems 89–92, each polynomial function has exactly one positive zero. Use the method of Example 10 to approximate the zero correct to two decimal places.

89. $f(x) = x^3 + x^2 + x - 4$
90. $f(x) = 2x^4 + x^2 - 1$
91. $f(x) = 2x^4 - 3x^3 - 4x^2 - 8$
92. $f(x) = 3x^3 - 2x^2 - 20$

Mixed Practice

In Problems 93–104, graph each polynomial function.

93. $f(x) = x^3 + 2x^2 - 5x - 6$
94. $f(x) = x^3 + 8x^2 + 11x - 20$
95. $f(x) = 2x^3 - x^2 + 2x - 1$
96. $f(x) = 2x^3 + x^2 + 2x + 1$
97. $f(x) = x^4 + x^2 - 2$
98. $f(x) = x^4 - 3x^2 - 4$
99. $f(x) = 4x^4 + 7x^2 - 2$
100. $f(x) = 4x^4 + 15x^2 - 4$
101. $f(x) = x^4 + x^3 - 3x^2 - x + 2$
102. $f(x) = x^4 - x^3 - 6x^2 + 4x + 8$
103. $f(x) = 4x^5 - 8x^4 - x + 2$
104. $f(x) = 4x^5 + 12x^4 - x - 3$
105. Suppose that $f(x) = 3x^3 + 16x^2 + 3x - 10$. Find the zeros of $f(x + 3)$.
106. Suppose that $f(x) = 4x^3 - 11x^2 - 26x + 24$. Find the zeros of $f(x - 2)$.

Applications and Extensions

107. Find k such that $f(x) = x^3 - kx^2 + kx + 2$ has the factor $x - 2$.
108. Find k such that $f(x) = x^4 - kx^3 + kx^2 + 1$ has the factor $x + 2$.
109. What is the remainder when $f(x) = 2x^{20} - 8x^{10} + x - 2$ is divided by $x - 1$?
110. What is the remainder when $f(x) = -3x^{17} + x^9 - x^5 + 2x$ is divided by $x + 1$?
111. Use the Factor Theorem to prove that $x - c$ is a factor of $x^n - c^n$ for any positive integer n.
112. Use the Factor Theorem to prove that $x + c$ is a factor of $x^n + c^n$ if $n \ge 1$ is an odd integer.
113. One solution of the equation $x^3 - 8x^2 + 16x - 3 = 0$ is 3. Find the sum of the remaining solutions.
114. One solution of the equation $x^3 + 5x^2 + 5x - 2 = 0$ is -2. Find the sum of the remaining solutions.
115. **Geometry** What is the length of the edge of a cube if, after a slice 1 inch thick is cut from one side, the volume remaining is 294 cubic inches?
116. **Geometry** What is the length of the edge of a cube if its volume could be doubled by an increase of 6 centimeters in one edge, an increase of 12 centimeters in a second edge, and a decrease of 4 centimeters in the third edge?
117. Let $f(x)$ be a polynomial function whose coefficients are integers. Suppose that r is a real zero of f and that the leading coefficient of f is 1. Use the Rational Zeros Theorem to show that r is either an integer or an irrational number.
118. Prove the Rational Zeros Theorem.

[**Hint:** Let $\frac{p}{q}$, where p and q have no common factors except 1 and -1, be a zero of the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

whose coefficients are all integers. Show that

$$a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0$$

Now, because p is a factor of the first n terms of this equation, p must also be a factor of the term $a_0 q^n$. Since p is not a factor of q (why?), p must be a factor of a_0. Similarly, q must be a factor of a_n.]

119. Bisection Method for Approximating Zeros of a Function f We begin with two consecutive integers, a and $a + 1$, such that $f(a)$ and $f(a + 1)$ are of opposite sign. Evaluate f at the midpoint m_1 of a and $a + 1$. If $f(m_1) = 0$, then m_1 is the zero of f, and we are finished. Otherwise, $f(m_1)$ is of opposite sign to either $f(a)$ or $f(a + 1)$. Suppose that it is $f(a)$ and $f(m_1)$ that are of opposite sign. Now evaluate f at the midpoint m_2 of a and m_1. Repeat this process until the desired degree of accuracy is obtained. Note that each iteration places the zero in an interval whose length is half that of the previous interval. Use the bisection method to approximate the zero of $f(x) = 8x^4 - 2x^2 + 5x - 1$ in the interval $[0, 1]$ correct to three decimal places.

[**Hint:** The process ends when both endpoints agree to the desired number of decimal places.]

Discussion and Writing

120. Is $\frac{1}{3}$ a zero of $f(x) = 2x^3 + 3x^2 - 6x + 7$? Explain.

121. Is $\frac{1}{3}$ a zero of $f(x) = 4x^3 - 5x^2 - 3x + 1$? Explain.

122. Is $\frac{3}{5}$ a zero of $f(x) = 2x^6 - 5x^4 + x^3 - x + 1$? Explain.

123. Is $\frac{2}{3}$ a zero of $f(x) = x^7 + 6x^5 - x^4 + x + 2$? Explain.

Retain Your Knowledge

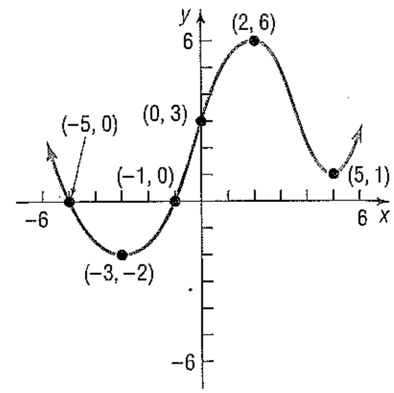
Problems 124–127 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

124. Solve $2x - 5y = 3$ for y.

125. Express the inequality $3 \le x < 8$ using interval notation.

126. Find the intercepts to the graph of the equation $3x + y^2 = 12$.

127. Use the figure to determine the interval (s) on which the function is increasing.



'Are You Prepared?' Answers

1. 3 **2.** $(3x + 2)(2x - 1)$ **3.** Quotient: $3x^3 + 4x^2 + 12x + 43$; Remainder: 125 **4.** $\frac{-1 - \sqrt{13}}{2}, \frac{-1 + \sqrt{13}}{2}$

3.3 Complex Zeros; Fundamental Theorem of Algebra

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Complex Numbers (Appendix A, Section A.11, pp. A89–A94)
- Complex Zeros of a Quadratic Function (Section 2.7, pp. 175–177)

Now Work the 'Are You Prepared?' problems on page 230.

OBJECTIVES
1 Use the Conjugate Pairs Theorem (p. 227)
2 Find a Polynomial Function with Specified Zeros (p. 228)
3 Find the Complex Zeros of a Polynomial Function (p. 229)

In Section 2.3, we found the real zeros of a quadratic function. That is, we found the real zeros of a polynomial function of degree 2. Then, in Section 2.7, we found the complex zeros of a quadratic function. That is, we found the complex zeros of a polynomial function of degree 2.

In Section 3.2, we found the real zeros of polynomial functions of degree 3 or higher. In this section we will find the *complex zeros* of polynomial functions of degree 3 or higher.