power function $y = 2x^5$, and the denominator can be approximated by the power function $y = x^3$. This means that as $x \to -\infty$ or as $x \to \infty$,

$$G(x) = \frac{2x^5 - x^3 + 2}{x^3 - 1} \approx \frac{2x^5}{x^3} = 2x^{5-3} = 2x^2$$

Since this is not linear, the graph of G has no horizontal or oblique asymptote. The graph of G will behave like $y = 2x^2$ as $x \to \pm\infty$.

Now Work PROBLEMS 43, 45 AND 47

3.4 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. ***True or False*** The quotient of two polynomial expressions is a rational expression. (p. A45)
2. What is the quotient and remainder when $3x^4 - x^2$ is divided by $x^3 - x^2 + 1$. (pp. A28–A29)
3. Graph $y = \frac{1}{x}$. (pp. 15–16)
4. Graph $y = 2(x + 1)^2 - 3$ using transformations. (pp. 89–93)

Concepts and Vocabulary

5. ***True or False*** The domain of every rational function is the set of all real numbers.
6. If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a __________ __________ of the graph of R.
7. If, as x approaches some number c, the values of $|R(x)| \to \infty$, then the line $x = c$ is a __________ __________ of the graph of R.
8. For a rational function R, if the degree of the numerator is less than the degree of the denominator, then R is __________.
9. ***True or False*** The graph of a rational function may intersect a horizontal asymptote.
10. ***True or False*** The graph of a rational function may intersect a vertical asymptote.
11. If a rational function is proper, then __________ is a horizontal asymptote.
12. ***True or False*** If the degree of the numerator of a rational function equals the degree of the denominator, then the ratio of the leading coefficients gives rise to the horizontal asymptote.

Skill Building

In Problems 13–24, find the domain of each rational function.

13. $R(x) = \frac{4x}{x - 3}$

14. $R(x) = \frac{5x^2}{3 + x}$

15. $H(x) = \frac{-4x^2}{(x - 2)(x + 4)}$

16. $G(x) = \frac{6}{(x + 3)(4 - x)}$

17. $F(x) = \frac{3x(x - 1)}{2x^2 - 5x - 3}$

18. $Q(x) = \frac{-x(1 - x)}{3x^2 + 5x - 2}$

19. $R(x) = \frac{x}{x^3 - 8}$

20. $R(x) = \frac{x}{x^4 - 1}$

21. $H(x) = \frac{3x^2 + x}{x^2 + 4}$

22. $G(x) = \frac{x - 3}{x^4 + 1}$

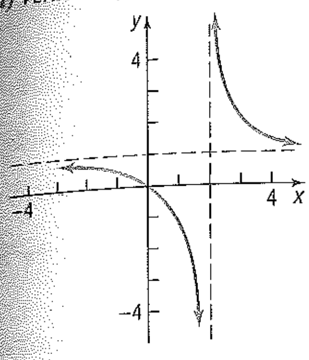
23. $R(x) = \frac{3(x^2 - x - 6)}{4(x^2 - 9)}$

24. $F(x) = \frac{-2(x^2 - 4)}{3(x^2 + 4x + 4)}$

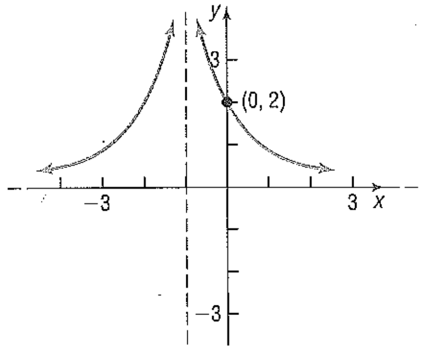
In Problems 25–30, use the graph shown to find:

(a) The domain and range of each function *(b) The intercepts, if any* *(c) Horizontal asymptotes, if any*
(d) Vertical asymptotes, if any *(e) Oblique asymptotes, if any*

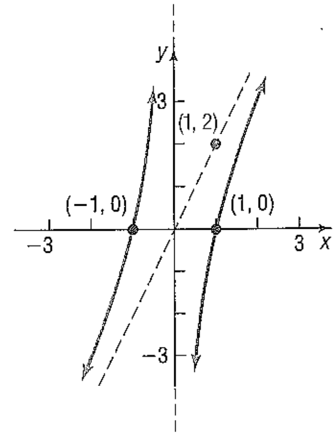
25.



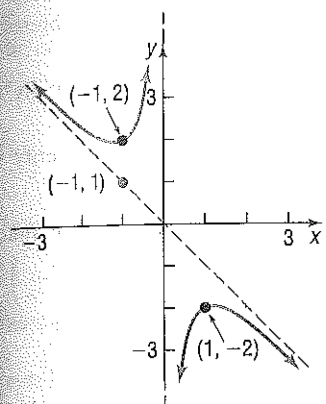
26.



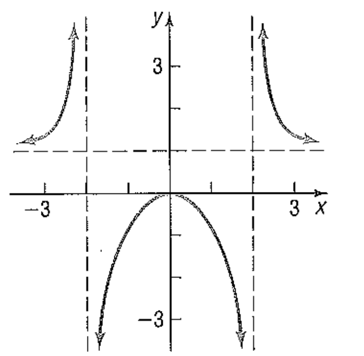
27.



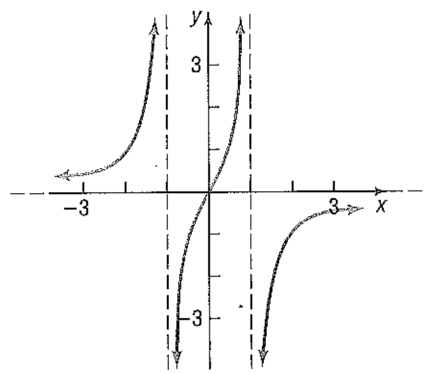
28.



29.



30.



In Problems 31–42, graph each rational function using transformations.

31. $F(x) = 2 + \dfrac{1}{x}$

32. $Q(x) = 3 + \dfrac{1}{x^2}$

33. $R(x) = \dfrac{1}{(x-1)^2}$

34. $R(x) = \dfrac{3}{x}$

35. $H(x) = \dfrac{-2}{x+1}$

36. $G(x) = \dfrac{2}{(x+2)^2}$

37. $R(x) = \dfrac{-1}{x^2+4x+4}$

38. $R(x) = \dfrac{1}{x-1} + 1$

39. $G(x) = 1 + \dfrac{2}{(x-3)^2}$

40. $F(x) = 2 - \dfrac{1}{x+1}$

41. $R(x) = \dfrac{x^2-4}{x^2}$

42. $R(x) = \dfrac{x-4}{x}$

In Problems 43–54, find the vertical, horizontal, and oblique asymptotes, if any, of each rational function.

43. $R(x) = \dfrac{3x}{x+4}$

44. $R(x) = \dfrac{3x+5}{x-6}$

45. $H(x) = \dfrac{x^3-8}{x^2-5x+6}$

46. $G(x) = \dfrac{x^3+1}{x^2-5x-14}$

47. $T(x) = \dfrac{x^3}{x^4-1}$

48. $P(x) = \dfrac{4x^2}{x^3-1}$

49. $Q(x) = \dfrac{2x^2-5x-12}{3x^2-11x-4}$

50. $F(x) = \dfrac{x^2+6x+5}{2x^2+7x+5}$

51. $R(x) = \dfrac{6x^2+7x-5}{3x+5}$

52. $R(x) = \dfrac{8x^2+26x-7}{4x-1}$

53. $G(x) = \dfrac{x^4-1}{x^2-x}$

54. $F(x) = \dfrac{x^4-16}{x^2-2x}$

Mixed Practice

In Problems 55–60, for each rational function:

(a) Graph the rational function using transformations. *(b) State the domain and the range.*
(c) State the vertical asymptote and the horizontal asymptote.

55. $R(x) = \dfrac{1}{x+3} - 2$

56. $R(x) = \dfrac{1}{x-1} + 5$

57. $R(x) = \dfrac{-2}{(x-1)^2} + 3$

58. $R(x) = \dfrac{3}{(x+2)^2} - 1$

59. $R(x) = 1 + \dfrac{4}{x^2-2x+1}$

60. $R(x) = 2 - \dfrac{2}{x^2+6x+9}$

Applications and Extensions

61. Gravity In physics, it is established that the acceleration due to gravity, g (in meters/sec^2), at a height h meters above sea level is given by

$$g(h) = \frac{3.99 \times 10^{14}}{(6.374 \times 10^6 + h)^2}$$

where 6.374×10^6 is the radius of Earth in meters.

(a) What is the acceleration due to gravity at sea level?

(b) The Willis Tower in Chicago, Illinois, is 443 meters tall. What is the acceleration due to gravity at the top of the Willis Tower?

(c) The peak of Mount Everest is 8848 meters above sea level. What is the acceleration due to gravity on the peak of Mount Everest?

(d) Find the horizontal asymptote of $g(h)$.

(e) Solve $g(h) = 0$. How do you interpret your answer?

62. Population Model A rare species of insect was discovered in the Amazon Rain Forest. Environmentalists protect the species by declaring the insect endangered and transplant the insect into a protected area. The population P of the insect t months after being transplanted is

$$P(t) = \frac{50(1 + 0.5t)}{(2 + 0.01t)}$$

(a) How many insects were discovered? In other words, what was the population when $t = 0$?

(b) What will the population be after 5 years?

(c) Determine the horizontal asymptote of $P(t)$. What is the largest population that the protected area can sustain?

63. Resistance in Parallel Circuits From Ohm's law for circuits, it follows that the total resistance R_{tot} of two components hooked in parallel is given by the equation

$$R_{tot} = \frac{R_1 R_2}{R_1 + R_2}$$

where R_1 and R_2 are the individual resistances.

(a) Let $R_1 = 10$ ohms, and graph R_{tot} as a function of R_2.

(b) Find and interpret any asymptotes of the graph obtained in part (a).

(c) If $R_2 = 2\sqrt{R_1}$, what value of R_1 will yield an R_{tot} of 17 ohms?

Source: en.wikipedia.org/wiki/Series_and_parallel_circuits

64. Newton's Method In calculus, you will learn that if

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is a polynomial function, then the *derivative* of $p(x)$ is

$$p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2 x + a_1$$

Newton's Method is an efficient method for finding the x-intercepts (or real zeros) of a function, such as $p(x)$. The following steps outline Newton's Method.

STEP 1: Select an initial value x_0 that is somewhat close to the x-intercept being sought.

STEP 2: Find values for x using the relation

$$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)} \quad n = 0, 1, 2, \ldots$$

until you get two consecutive values x_n and x_{n+1} that agree to whatever decimal place accuracy you desire.

STEP 3: The approximate zero will be x_{n+1}.

Consider the polynomial $p(x) = x^3 - 7x - 40$.

(a) Evaluate $p(5)$ and $p(-3)$.

(b) What might we conclude about a zero of p? Explain.

(c) Use Newton's Method to approximate an x-intercept, r, $-3 < r < 5$, of $p(x)$ to four decimal places.

(d) Use a graphing utility to graph $p(x)$ and verify your answer in part (c).

(e) Using a graphing utility, evaluate $p(r)$ to verify your result.

65. Exploration The standard form of the rational function $R(x) = \frac{mx + b}{cx + d}$ where $c \neq 0$, is $R(x) = a \cdot \left(\frac{1}{x - h}\right) + k$. To write a rational function in standard form requires long division.

(a) Write the rational function $R(x) = \frac{2x + 3}{x - 1}$ in standard form by writing R in the form

$$\text{Quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

(b) Graph R using transformations.

(c) Determine the vertical asymptote and the horizontal asymptote of R.

66. Exploration See problem 65.

(a) Write the rational function $R(x) = \frac{-6x + 16}{2x - 7}$ in the form Quotient $+ \frac{\text{remainder}}{\text{divisor}}$.

(b) Factor out the coefficient on x in the divisor to write R in the form $R(x) = \frac{\text{remainder}}{m(x - h)} + k$.

(c) Write the function found in part (b) in the standard form $R(x) = a \cdot \left(\frac{1}{x - h}\right) + k$.

(d) Graph the function found in part (c) using transformations.

(e) Determine the vertical asymptote and the horizontal asymptote of R.

Discussion and Writing

67. If the graph of a rational function R has the vertical asymptote $x = 4$, the factor $x - 4$ must be present in the denominator of R. Explain why.

68. If the graph of a rational function R has the horizontal asymptote $y = 2$, the degree of the numerator of R equals the degree of the denominator of R. Explain why.

69. Can the graph of a rational function have both a horizontal and an oblique asymptote? Explain.

70. Make up a rational function that has $y = 2x + 1$ as an oblique asymptote. Explain the methodology that you used.

Retain Your Knowledge

Problems 71–74 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

71. Find the equations of a vertical line passing through the point $(5, -3)$.

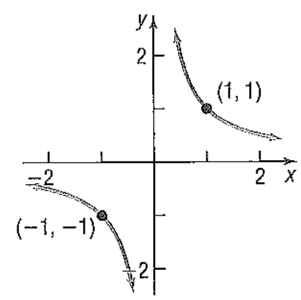
72. Solve: $\frac{2}{5}(3x - 7) + 1 = \frac{x}{4} - 2$

73. Determine whether the graph of the equation $2x^3 - xy^2 = 4$ is symmetric with respect to the x-axis, the y-axis, the origin, or none of these.

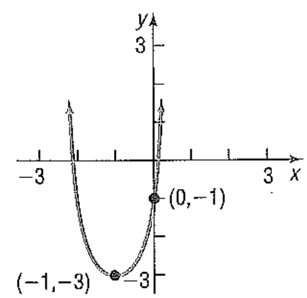
74. What are the points of intersection of the graphs of the functions $f(x) = -3x + 2$ and $g(x) = x^2 - 2x - 4$?

'Are You Prepared?' Answers

1. True **2.** Quotient: $3x + 3$; remainder: $2x^2 - 3x - 3$ **3.**



4.



3.5 The Graph of a Rational Function

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Intercepts (Foundations, Section 2, pp. 11–12)
- Solving Rational Equations (Appendix A, Section A.8, pp. A66–A67)

Now Work the 'Are You Prepared?' problems on page 255.

OBJECTIVES 1 Analyze the Graph of a Rational Function (p. 243)
2 Solve Applied Problems Involving Rational Functions (p. 254)

1 Analyze the Graph of a Rational Function

We commented earlier that calculus provides the tools required to graph a polynomial function accurately. The same holds true for rational functions. However, we can gather together quite a bit of information about their graphs to get an idea of the general shape and position of the graph.

EXAMPLE 1 How to Analyze the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \frac{x - 1}{x^2 - 4}$

Step-by-Step Solution

Step 1: Factor the numerator and denominator of R. Find the domain of the rational function.

$$R(x) = \frac{x - 1}{x^2 - 4} = \frac{x - 1}{(x + 2)(x - 2)}$$

The domain of R is $\{x \mid x \neq -2, x \neq 2\}$.

Step 2: Write R in lowest terms.

Because there are no common factors between the numerator and denominator, R is in lowest terms.