SUMMARY

Analyzing the Graph of a Rational Function R

STEP 1: Factor the numerator and denominator of R. Find the domain of the rational function.

STEP 2: Write R in lowest terms.

STEP 3: Locate the intercepts of the graph. Determine the behavior of the graph of R at each x-intercept.

STEP 4: Determine the vertical asymptotes. Graph each vertical asymptote using a dashed line. Determine the behavior of the graph of R on either side of each vertical asymptote.

STEP 5: Determine the horizontal or oblique asymptote, if one exists. Determine points, if any, at which the graph of R intersects this asymptote. Graph the asymptote using a dashed line. Plot any points at which the graph of R intersects the asymptote.

STEP 6: Use the zeros of the numerator and denominator of R to divide the x-axis into intervals. Determine where the graph of R is above or below the x-axis by choosing a number in each interval and evaluating R there. Plot the points found.

STEP 7: Use the results obtained in Steps 1 through 6 to graph R.

EXAMPLE 2 Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{x^2 - 1}{x}$

Solution

STEP 1: $R(x) = \dfrac{(x+1)(x-1)}{x}$. The domain of R is $\{x \mid x \neq 0\}$.

STEP 2: R is in lowest terms.

STEP 3: Because x cannot equal 0, there is no y-intercept. The graph has tw x-intercepts, -1 and 1, each with odd multiplicity. Plot the points $(-1, ($ and $(1, 0)$. The graph will cross the x-axis at both points.

STEP 4: The real zero of the denominator with R in lowest terms is 0, so the graph c R has the line $x = 0$ (the y-axis) as a vertical asymptote. Graph $x = 0$ usin a dashed line. The multiplicity of 0 is odd, so the graph will approach ∞ o one side of the asymptote $x = 0$, and $-\infty$ on the other.

STEP 5: Since the degree of the numerator, 2, is one greater than the degree of th denominator, 1, the rational function will have an oblique asymptote. T find the oblique asymptote, use long division.

$$\begin{array}{r} x \\ x\overline{)x^2 - 1} \\ \underline{x^2 } \\ -1 \end{array}$$

The quotient is x, so the line $y = x$ is an oblique asymptote of the grapl Graph $y = x$ using a dashed line.

To determine whether the graph of R intersects the asymptote $y =$: solve the equation $R(x) = x$.

$$\begin{aligned} R(x) = \frac{x^2 - 1}{x} &= x \\ x^2 - 1 &= x^2 \\ -1 &= 0 \quad \text{Impossible} \end{aligned}$$

The equation $\dfrac{x^2 - 1}{x} = x$ has no solution, so the graph of R does n intersect the line $y = x$.

NOTE Because the denominator of the rational function is a monomial, we can also find the oblique asymptote as follows:

$$\frac{x^2 - 1}{x} = \frac{x^2}{x} - \frac{1}{x} = x - \frac{1}{x}$$

Since $\dfrac{1}{x} \to 0$ as $x \to \infty$, $y = x$ is the oblique asymptote. ■

Step 6: The zeros of the numerator are -1 and 1; the zero of the denominator is 0. Use these values to divide the x-axis into four intervals:

$$(-\infty, -1) \qquad (-1, 0) \qquad (0, 1) \qquad (1, \infty)$$

Now construct Table 13. Plot the points from Table 13. You should now have Figure 35(a).

Table 13

	-1	0	1	x
Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Number chosen	-2	$-\frac{1}{2}$	$\frac{1}{2}$	2
Value of R	$R(-2) = -\frac{3}{2}$	$R\left(-\frac{1}{2}\right) = \frac{3}{2}$	$R\left(\frac{1}{2}\right) = -\frac{3}{2}$	$R(2) = \frac{3}{2}$
Location of graph	Below x-axis	Above x-axis	Below x-axis	Above x-axis
Point on graph	$\left(-2, -\frac{3}{2}\right)$	$\left(-\frac{1}{2}, \frac{3}{2}\right)$	$\left(\frac{1}{2}, -\frac{3}{2}\right)$	$\left(2, \frac{3}{2}\right)$

Step 7: The graph crosses the x-axis at $x = -1$ and $x = 1$, changing from being below the x-axis to being above it in both cases.

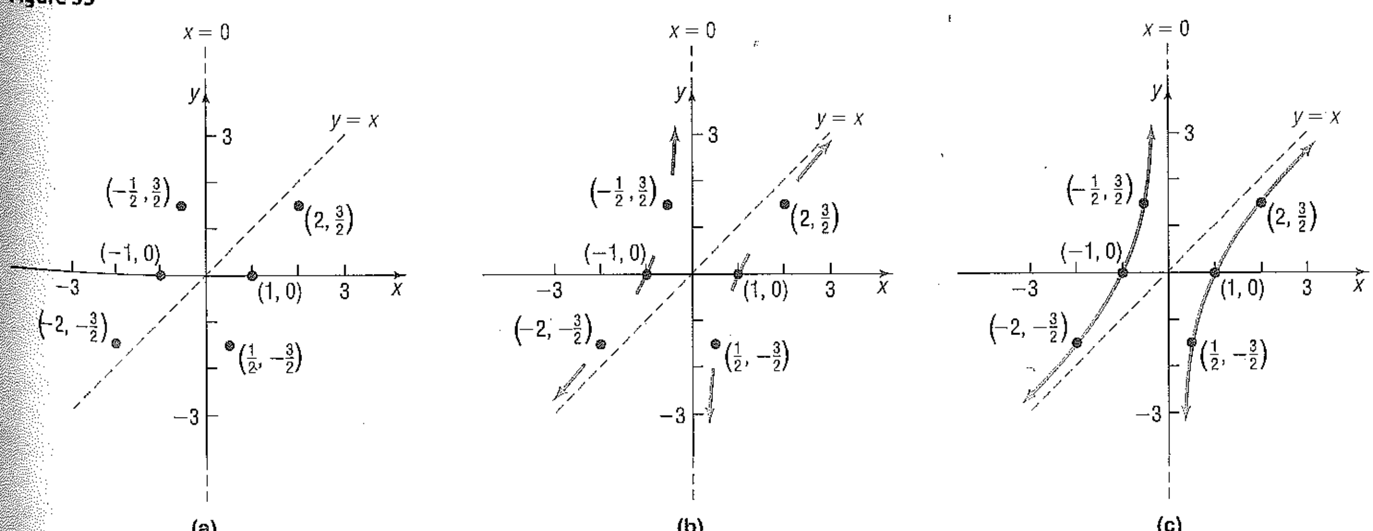
Since the graph of R is below the x-axis for $x < -1$ and is above the x-axis for $x > 1$, and since the graph of R does not intersect the oblique asymptote $y = x$, the graph of R will approach the line $y = x$ as shown in Figure 35(b).

Since the graph of R is above the x-axis for $-1 < x < 0$, the graph of R will approach the vertical asymptote $x = 0$ at the top to the left of $x = 0$ $\left[\lim_{x \to 0^-} R(x) = \infty\right]$; since the graph of R approaches ∞ on one side of the asymptote and $-\infty$ on the other, the graph of R will approach the vertical asymptote $x = 0$ at the bottom to the right of $x = 0$ $\left[\lim_{x \to 0^+} R(x) = -\infty\right]$. See Figure 35(b).

The complete graph is given in Figure 35(c).

NOTE Notice that R in Example 2 is an odd function. Do you see the symmetry about the origin in the graph of R in Figure 35(c)?

Figure 35



Seeing the Concept

Graph $R(x) = \dfrac{x^2 - 1}{x}$ and compare what you see with Figure 35(c). Could you have predicted from the graph that $y = x$ is an oblique asymptote? Graph $y = x$ and ZOOM-OUT. What do you observe?

Now Work PROBLEM 15

EXAMPLE 3 **Analyzing the Graph of a Rational Function**

Analyze the graph of the rational function: $R(x) = \dfrac{x^4 + 1}{x^2}$

Solution

STEP 1: R is completely factored. The domain of R is $\{x \mid x \neq 0\}$.

STEP 2: R is in lowest terms.

STEP 3: There is no y-intercept. Since $x^4 + 1 = 0$ has no real solutions, there are no x-intercepts.

STEP 4: R is in lowest terms, so $x = 0$ (the y-axis) is a vertical asymptote of R. Graph the line $x = 0$ using dashes. The multiplicity of 0 is even, so the graph will approach either ∞ or $-\infty$ on both sides of the asymptote.

STEP 5: Since the degree of the numerator, 4, is two more than the degree of the denominator, 2, the rational function will not have a horizontal or oblique asymptote. Find the end behavior of R. As $|x| \to \infty$,

$$R(x) = \frac{x^4 + 1}{x^2} \approx \frac{x^4}{x^2} = x^2$$

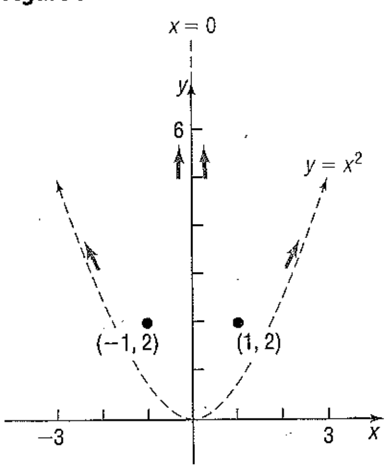
The graph of R will approach the graph of $y = x^2$ as $x \to -\infty$ and as $x \to \infty$. The graph of R does not intersect $y = x^2$. Do you know why? Graph $y = x^2$ using dashes.

STEP 6: The numerator has no real zeros, and the denominator has one real zero at 0. Divide the x-axis into the two intervals

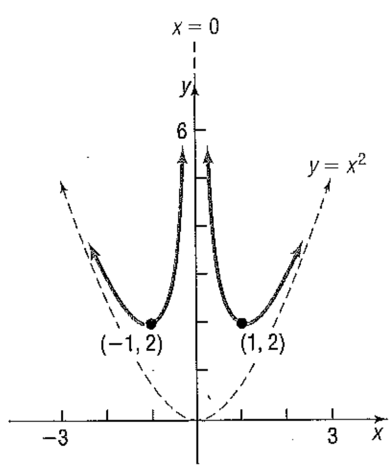
$$(-\infty, 0) \qquad (0, \infty)$$

and construct Table 14.

Figure 36



(a)



(b)

NOTE Notice that R in Example 3 is an even function. Do you see the symmetry about the y-axis in the graph of R? ■

Table 14

	0 → x	
Interval	$(-\infty, 0)$	$(0, \infty)$
Number chosen	-1	1
Value of R	$R(-1) = 2$	$R(1) = 2$
Location of graph	Above x-axis	Above x-axis
Point on graph	$(-1, 2)$	$(1, 2)$

STEP 7: Since the graph of R is above the x-axis and does not intersect $y = x^2$, place arrows above $y = x^2$ as shown in Figure 36(a). Also, since the graph of R is above the x-axis, and the multiplicity of the zero that gives rise to the vertical asymptote, $x = 0$, is even, it will approach the vertical asymptote $x = 0$ at the top to the left of $x = 0$ and at the top to the right of $x = 0$. See Figure 36(a).

Figure 36(b) shows the complete graph.

Seeing the Concept

Graph $R(x) = \dfrac{x^4 + 1}{x^2}$ and compare what you see with Figure 36(b). Use MINIMUM to find the two turning points. Enter $Y_2 = x^2$ and ZOOM-OUT. What do you see?

Now Work PROBLEM 13

EXAMPLE 4 Analyzing the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

Solution

STEP 1: Factor R to get

$$R(x) = \frac{3x(x - 1)}{(x + 4)(x - 3)}$$

The domain of R is $\{x \mid x \neq -4, x \neq 3\}$.

STEP 2: R is in lowest terms.

STEP 3: The y-intercept is $R(0) = 0$. Plot the point $(0, 0)$. Since the real solutions of the equation $3x(x - 1) = 0$ are $x = 0$ and $x = 1$, the graph has two x-intercepts, 0 and 1, each with odd multiplicity. Plot the points $(0, 0)$ and $(1, 0)$. The graph will cross the x-axis at both points.

STEP 4: R is in lowest terms. The real solutions of the equation $(x + 4)(x - 3) = 0$ are $x = -4$ and $x = 3$, so the graph of R has two vertical asymptotes, the lines $x = -4$ and $x = 3$. Graph these lines using dashes. The multiplicities that give rise to the vertical asymptotes are both odd, so the graph will approach ∞ on one side of each vertical asymptote and $-\infty$ on the other side.

STEP 5: Since the degree of the numerator equals the degree of the denominator, the graph has a horizontal asymptote. To find it, form the quotient of the leading coefficient of the numerator, 3, and the leading coefficient of the denominator, 1. The graph of R has the horizontal asymptote $y = 3$.

To find out whether the graph of R intersects the asymptote, solve the equation $R(x) = 3$.

$$R(x) = \frac{3x^2 - 3x}{x^2 + x - 12} = 3$$
$$3x^2 - 3x = 3x^2 + 3x - 36$$
$$-6x = -36$$
$$x = 6$$

The graph intersects the line $y = 3$ at $x = 6$, and $(6, 3)$ is a point on the graph of R. Plot the point $(6, 3)$ and graph the line $y = 3$ using dashes.

STEP 6: The real zeros of the numerator, 0 and 1, and the real zeros of the denominator, -4 and 3, divide the x-axis into five intervals:

$$(-\infty, -4) \quad (-4, 0) \quad (0, 1) \quad (1, 3) \quad (3, \infty)$$

Construct Table 15. See page 250. Plot the points from Table 15. Figure 37(a) shows the graph so far.

STEP 7: Since the graph of R is above the x-axis for $x < -4$ and only crosses the line $y = 3$ at $(6, 3)$, as x approaches $-\infty$ the graph of R will approach the horizontal asymptote $y = 3$ from above $\left(\lim_{x \to -\infty} R(x) = 3\right)$. The graph of R will approach the vertical asymptote $x = -4$ at the top to the left of $x = -4$ $\left(\lim_{x \to -4^-} R(x) = \infty\right)$ and at the bottom to the right of $x = -4$ $\left(\lim_{x \to -4^+} R(x) = -\infty\right)$. The graph of R will approach the vertical asymptote

Table 15

	-4	0	1	3	
Interval	$(-\infty, -4)$	$(-4, 0)$	$(0, 1)$	$(1, 3)$	$(3, \infty)$
Number chosen	-5	-2	$\frac{1}{2}$	2	4
Value of R	$R(-5) = 11.25$	$R(-2) = -1.8$	$R\left(\frac{1}{2}\right) = \frac{1}{15}$	$R(2) = -1$	$R(4) = 4.5$
Location of graph	Above x-axis	Below x-axis	Above x-axis	Below x-axis	Above x-axis
Point on graph	$(-5, 11.25)$	$(-2, -1.8)$	$\left(\frac{1}{2}, \frac{1}{15}\right)$	$(2, -1)$	$(4, 4.5)$

$x = 3$ at the bottom to the left of $x = 3$ $\left(\lim_{x \to 3^-} R(x) = -\infty\right)$ and at the top to the right of $x = 3$ $\left(\lim_{x \to 3^+} R(x) = \infty\right)$.

We do not know whether the graph of R crosses or touches the line $y = 3$ at $(6, 3)$. To see whether the graph, in fact, crosses or touches the line $y = 3$, plot an additional point to the right of $(6, 3)$. We use $x = 7$ to find $R(7) = \frac{63}{22} < 3$. The graph crosses $y = 3$ at $x = 6$. Because $(6, 3)$ is the only point where the graph of R intersects the asymptote $y = 3$, the graph must approach the line $y = 3$ from below as $x \to \infty$ $\left(\lim_{x \to \infty} R(x) = 3\right)$. The graph crosses the x-axis at $x = 0$, changing from being below the x-axis to being above. The graph also crosses the x-axis at $x = 1$, changing from being above the x-axis to being below. See Figure 37(b). The complete graph is shown in Figure 37(c).

Figure 37

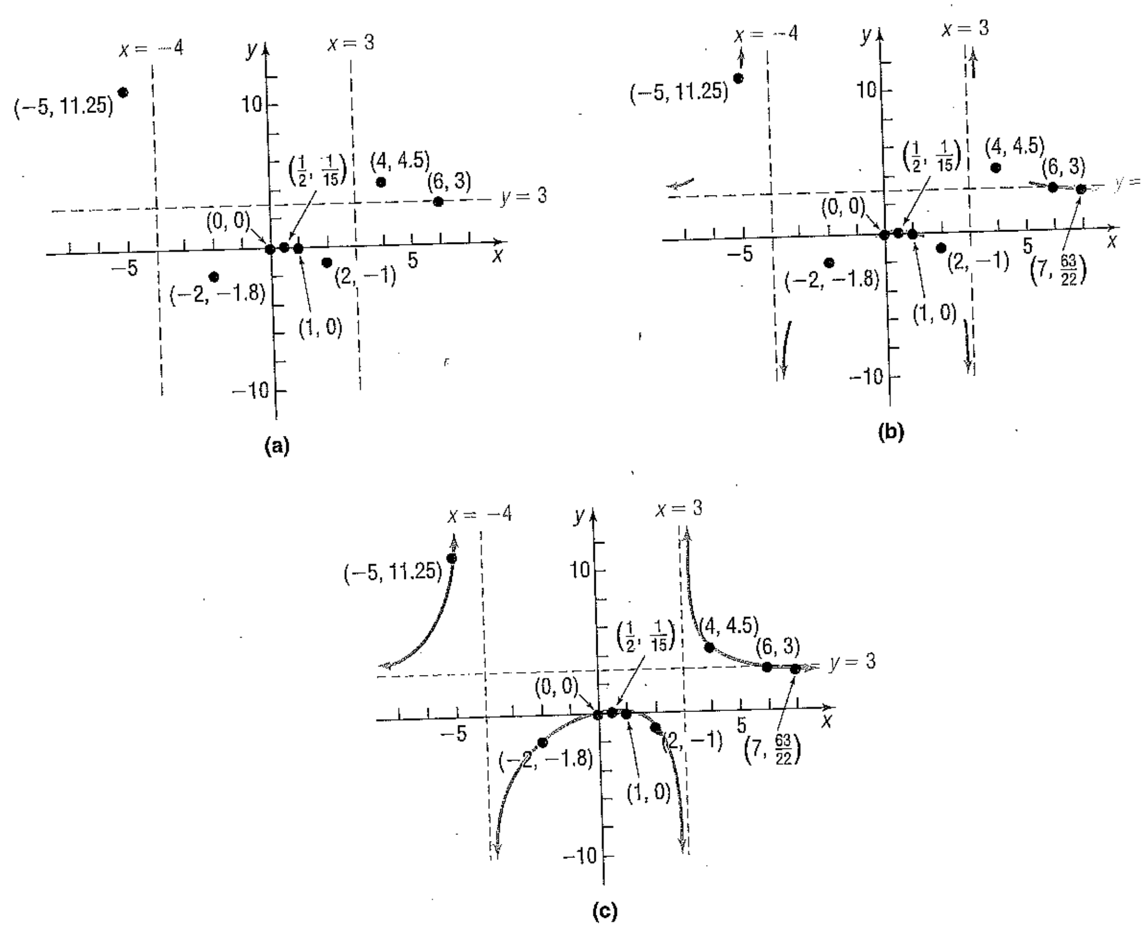
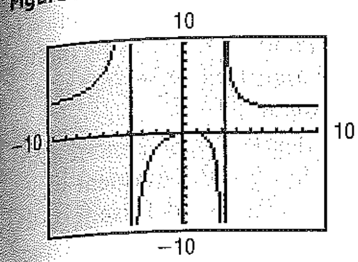


Figure 38



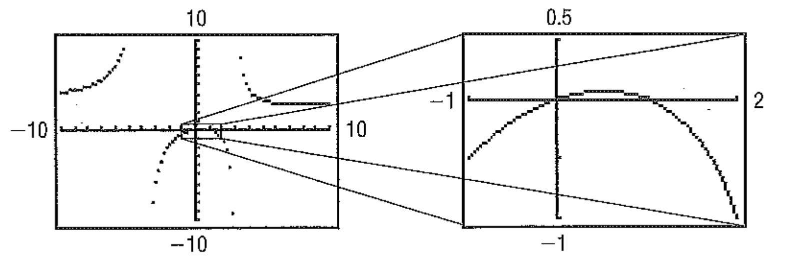
Connected mode

Exploration

Graph $R(x) = \dfrac{3x^2 - 3x}{x^2 + x - 12}$

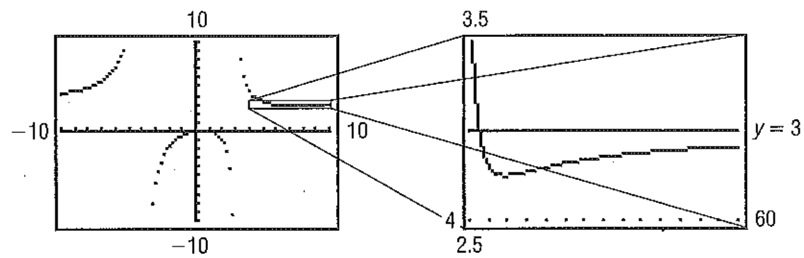
Result Figure 38 shows the graph in connected mode, and Figure 39(a) shows it in dot mode. Neither graph displays clearly the behavior of the function between the two x-intercepts, 0 and 1. Nor do they clearly display the fact that the graph crosses the horizontal asymptote at (6, 3). To see these parts better, graph R for $-1 \le x \le 2$ [Figure 39(b)] and for $4 \le x \le 60$ [Figure 40(b)].

Figure 39



Dot mode
(a) (b)

Figure 40



Dot mode
(a) (b)

The new graphs reflect the behavior produced by the analysis. Furthermore, we observe two turning points, one between 0 and 1 and the other to the right of 6. Rounded to two decimal places, these turning points are (0.52, 0.07) and (11.48, 2.75).

EXAMPLE 5 Analyzing the Graph of a Rational Function with a Hole

Analyze the graph of the rational function: $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$

Solution **Step 1:** Factor R and obtain

$$R(x) = \frac{(2x - 1)(x - 2)}{(x + 2)(x - 2)}$$

The domain of R is $\{x \mid x \neq -2, x \neq 2\}$.

Step 2: In lowest terms,

$$R(x) = \frac{2x - 1}{x + 2} \qquad x \neq -2, x \neq 2$$

Step 3: The y-intercept is $R(0) = -\frac{1}{2}$. Plot the point $\left(0, -\frac{1}{2}\right)$. The graph has one x-intercept, $\frac{1}{2}$, with odd multiplicity. Plot the point $\left(\frac{1}{2}, 0\right)$. The graph will cross the x-axis at $x = \frac{1}{2}$.

Step 4: Since $x + 2$ is the only factor of the denominator of $R(x)$ *in lowest terms*, the graph has one vertical asymptote, $x = -2$. However, the rational function is undefined at both $x = 2$ and $x = -2$. Graph the line $x = -2$ using dashes. The multiplicity of -2 is odd, so the graph will approach ∞ on one side of the vertical asymptote and $-\infty$ on the other side.

Step 5: Since the degree of the numerator equals the degree of the denominator, the graph has a horizontal asymptote. To find it, form the quotient of the leading coefficient of the numerator, 2, and the leading coefficient of the denominator, 1. The graph of R has the horizontal asymptote $y = 2$. Graph the line $y = 2$ using dashes.

To find out whether the graph of R intersects the horizontal asymptote $y = 2$, solve the equation $R(x) = 2$.

$$R(x) = \frac{2x - 1}{x + 2} = 2$$
$$2x - 1 = 2(x + 2)$$
$$2x - 1 = 2x + 4$$
$$-1 = 4 \quad \text{Impossible}$$

The graph does not intersect the line $y = 2$.

Step 6: Look at the factored expression for R in Step 1. The real zeros of the numerator and denominator, -2, $\frac{1}{2}$, and 2, divide the x-axis into four intervals:

$$(-\infty, -2) \quad \left(-2, \frac{1}{2}\right) \quad \left(\frac{1}{2}, 2\right) \quad (2, \infty)$$

Construct Table 16. Plot the points in Table 16.

Table 16

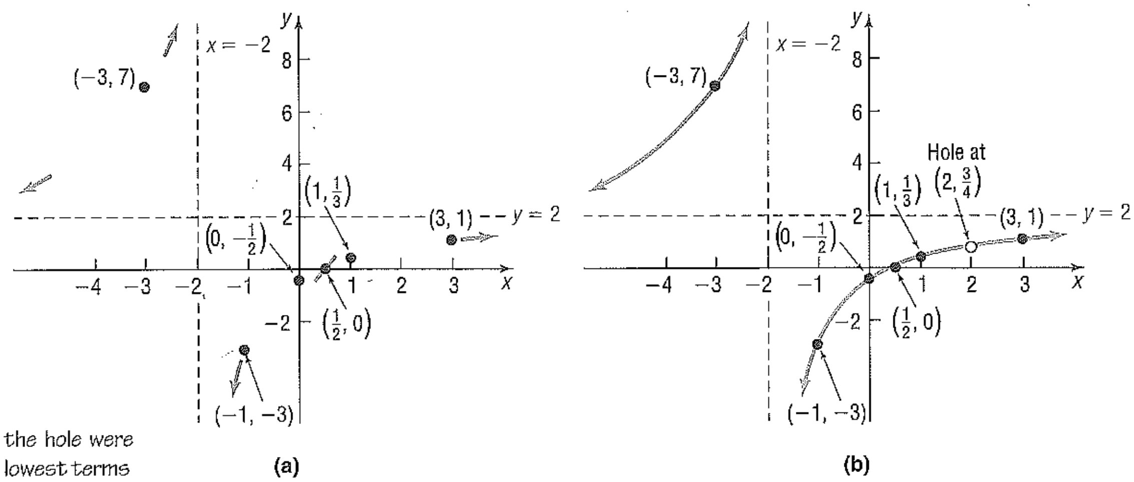
	-2	$1/2$	2	x
Interval	$(-\infty, -2)$	$\left(-2, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 2\right)$	$(2, \infty)$
Number chosen	-3	-1	1	3
Value of R	$R(-3) = 7$	$R(-1) = -3$	$R(1) = \frac{1}{3}$	$R(3) = 1$
Location of graph	Above x-axis	Below x-axis	Above x-axis	Above x-axis
Point on graph	$(-3, 7)$	$(-1, -3)$	$\left(1, \frac{1}{3}\right)$	$(3, 1)$

Step 7: From Table 16 we know that the graph of R is above the x-axis for $x < -$ From Step 5 we know that the graph of R does not intersect the asymptc $y = 2$. Therefore, the graph of R will approach $y = 2$ from above $x \to -\infty$ and will approach the vertical asymptote $x = -2$ at the top frc the left.

Since the graph of R is below the x-axis for $-2 < x < \frac{1}{2}$, the graph v approach $x = -2$ at the bottom from the right. Finally, since the graph o is above the x-axis for $x > \frac{1}{2}$ and does not intersect the horizontal asympt $y = 2$, the graph of R will approach $y = 2$ from below as $x \to \infty$. The gr crosses the x-axis at $x = \frac{1}{2}$, changing from being below the x-axis to be above. See Figure 41(a).

See Figure 41(b) for the complete graph. Since R is not defined at 2, there is a hole at the point $\left(2, \frac{3}{4}\right)$.

Figure 41



NOTE The coordinates of the hole were obtained by evaluating R in lowest terms at 2. R in lowest terms is $\frac{2x-1}{x+2}$, which, at $x = 2$, is $\frac{2(2)-1}{2+2} = \frac{3}{4}$.

Exploration

Graph $R(x) = \frac{2x^2 - 5x + 2}{x^2 - 4}$. Do you see the hole at $\left(2, \frac{3}{4}\right)$? TRACE along the graph. Did you obtain an ERROR at $x = 2$? Are you convinced that an algebraic analysis of a rational function is required in order to accurately interpret the graph obtained with a graphing utility?

As Example 5 shows, **the zeros of the denominator of a rational function give rise to either vertical asymptotes or holes in the graph.**

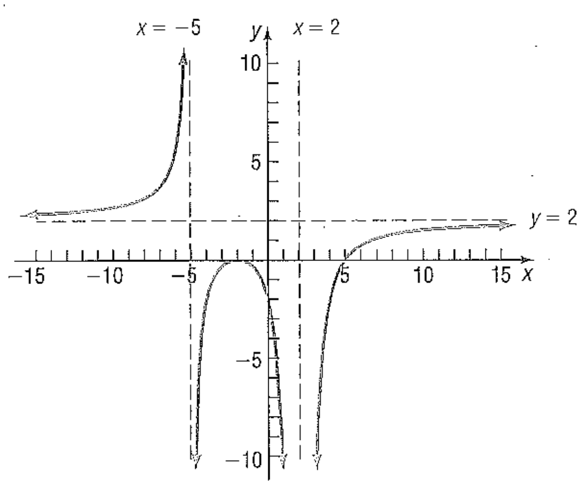
Now Work PROBLEM 33

In calculus, a discontinuity such as the hole in the graph of Example 5 is called a **removable** discontinuity. It is considered removable because the function can be appropriately redefined at the point of discontinuity to make the graph continuous at that point. Vertical asymptotes are examples of discontinuities that are **nonremovable.** See Problem 69.

EXAMPLE 6 **Constructing a Rational Function from Its Graph**

Find a rational function that might have the graph shown in Figure 42.

Figure 42



Solution The numerator of a rational function $R(x) = \frac{p(x)}{q(x)}$ in lowest terms determines the x-intercepts of its graph. The graph shown in Figure 42 has x-intercepts -2 (even multiplicity; graph touches the x-axis) and 5 (odd multiplicity; graph crosses the x-axis). So one possibility for the numerator is $p(x) = (x + 2)^2(x - 5)$.

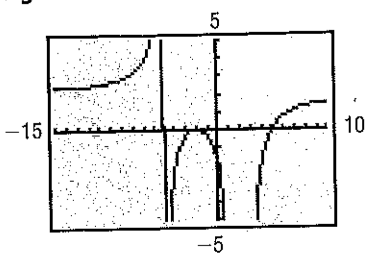
The denominator of a rational function in lowest terms determines the vertical asymptotes of its graph. The vertical asymptotes of the graph are $x = -5$ and $x = 2$. Since $R(x)$ approaches ∞ to the left of $x = -5$ and $R(x)$ approaches $-\infty$ to the right of $x = -5$, we know that $(x + 5)$ is a factor of odd multiplicity in $q(x)$. Also, $R(x)$ approaches $-\infty$ on both sides of $x = 2$, so $(x - 2)$ is a factor of even multiplicity in $q(x)$. A possibility for the denominator is $q(x) = (x + 5)(x - 2)^2$. So far we have

$$R(x) = \frac{(x + 2)^2(x - 5)}{(x + 5)(x - 2)^2}.$$

However, the horizontal asymptote of the graph given in Figure 42 is $y = 2$, so we know that the degree of the numerator must equal the degree of the denominator and that the quotient of leading coefficients must be $\frac{2}{1}$. This leads to

$$R(x) = \frac{2(x + 2)^2(x - 5)}{(x + 5)(x - 2)^2}$$

Figure 43



Check: Figure 43 shows the graph of R on a graphing utility. Since Figure 43 looks similar to Figure 42, we have found a rational function R for the graph in Figure 42.

Now Work PROBLEM 57

2 Solve Applied Problems Involving Rational Functions

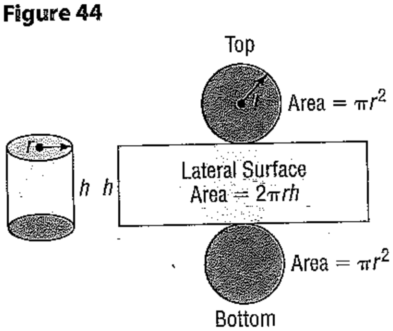
EXAMPLE 7 **Finding the Least Cost of a Can**

Reynolds Metal Company manufactures aluminum cans in the shape of a cylinder with a capacity of 500 cubic centimeters $\left(\frac{1}{2}\text{ liter}\right)$. The top and bottom of the can are made of a special aluminum alloy that costs 0.05¢ per square centimeter. The sides of the can are made of material that costs 0.02¢ per square centimeter.

(a) Express the cost of material for the can as a function of the radius r of the can.
(b) Use a graphing utility to graph the function $C = C(r)$.
(c) What value of r will result in the least cost?
(d) What is this least cost?

Solution (a) Figure 44 illustrates the components of a can in the shape of a right circular cylinder. Notice that the material required to produce a cylindrical can of height h and radius r consists of a rectangle of area $2\pi rh$ and two circles, each of area πr^2. The total cost C (in cents) of manufacturing the can is therefore

Figure 44

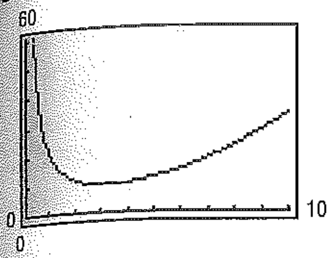


$$\begin{aligned} C &= \text{Cost of the top and bottom} + \text{Cost of the side} \\ &= \underbrace{2(\pi r^2)}_{\substack{\text{Total area}\\\text{of top and}\\\text{bottom}}} \cdot \underbrace{(0.05)}_{\substack{\text{Cost/unit}\\\text{area}}} + \underbrace{(2\pi rh)}_{\substack{\text{Total}\\\text{area of}\\\text{side}}} \cdot \underbrace{(0.02)}_{\substack{\text{Cost/unit}\\\text{area}}} \\ &= 0.10\pi r^2 + 0.04\pi rh \end{aligned}$$

There is an additional restriction that the height h and radius r must be chosen so that the volume V of the can is 500 cubic centimeters. Since $V = \pi r^2 h$, we have

$$500 = \pi r^2 h \quad \text{so} \quad h = \frac{500}{\pi r^2}$$

Figure 45



Substituting this expression for h, the cost C, in cents, as a function of the radius r is

$$C(r) = 0.10\pi r^2 + 0.04\pi r \cdot \frac{500}{\pi r^2} = 0.10\pi r^2 + \frac{20}{r} = \frac{0.10\pi r^3 + 20}{r}$$

(b) See Figure 45 for the graph of $C = C(r)$.

(c) Using the MINIMUM command, the cost is least for a radius of about 3.17 centimeters.

(d) The least cost is $C(3.17) \approx 9.47¢$.

Now Work PROBLEM 67

3.5 Assess Your Understanding

'Are You Prepared?' *The answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Find the intercepts of the graph of the equation $y = \frac{x^2 - 1}{x^2 - 4}$. (pp. 11–12)

2. Solve $\frac{x - 3}{x^2 + 1} = -2$. (pp. A66–A67)

Concepts and Vocabulary

3. If the numerator and the denominator of a rational function have no common factors, the rational function is ________.

4. The graph of a rational function never intersects a ________ asymptote.

5. ***True or False*** The graph of a rational function sometimes intersects an oblique asymptote.

6. ***True or False*** The graph of a rational function sometimes has a hole.

Skill Building

In Problems 7–50, follow Steps 1 through 7 on page 246 to analyze the graph of each function.

7. $R(x) = \frac{x + 1}{x(x + 4)}$

8. $R(x) = \frac{x}{(x - 1)(x + 2)}$

9. $R(x) = \frac{3x + 3}{2x + 4}$

10. $R(x) = \frac{2x + 4}{x - 1}$

11. $R(x) = \frac{3}{x^2 - 4}$

12. $R(x) = \frac{6}{x^2 - x - 6}$

13. $P(x) = \frac{x^4 + x^2 + 1}{x^2 - 1}$

14. $Q(x) = \frac{x^4 - 1}{x^2 - 4}$

15. $H(x) = \frac{x^3 - 1}{x^2 - 9}$

16. $G(x) = \frac{x^3 + 1}{x^2 + 2x}$

17. $R(x) = \frac{x^2}{x^2 + x - 6}$

18. $R(x) = \frac{x^2 + x - 12}{x^2 - 4}$

19. $G(x) = \frac{x}{x^2 - 4}$

20. $G(x) = \frac{3x}{x^2 - 1}$

21. $R(x) = \frac{3}{(x - 1)(x^2 - 4)}$

22. $R(x) = \frac{-4}{(x + 1)(x^2 - 9)}$

23. $H(x) = \frac{x^2 - 1}{x^4 - 16}$

24. $H(x) = \frac{x^2 + 4}{x^4 - 1}$

25. $F(x) = \frac{x^2 - 3x - 4}{x + 2}$

26. $F(x) = \frac{x^2 + 3x + 2}{x - 1}$

27. $R(x) = \frac{x^2 + x - 12}{x - 4}$

28. $R(x) = \frac{x^2 - x - 12}{x + 5}$

29. $F(x) = \frac{x^2 + x - 12}{x + 2}$

30. $G(x) = \frac{x^2 - x - 12}{x + 1}$

31. $R(x) = \frac{x(x - 1)^2}{(x + 3)^3}$

32. $R(x) = \frac{(x - 1)(x + 2)(x - 3)}{x(x - 4)^2}$

33. $R(x) = \frac{x^2 + x - 12}{x^2 - x - 6}$

34. $R(x) = \frac{x^2 + 3x - 10}{x^2 + 8x + 15}$

35. $R(x) = \frac{6x^2 - 7x - 3}{2x^2 - 7x + 6}$

36. $R(x) = \frac{8x^2 + 26x + 15}{2x^2 - x - 15}$

37. $R(x) = \frac{x^2 + 5x + 6}{x + 3}$

38. $R(x) = \frac{x^2 + x - 30}{x + 6}$

39. $H(x) = \frac{3x - 6}{4 - x^2}$

40. $H(x) = \dfrac{2 - 2x}{x^2 - 1}$

41. $F(x) = \dfrac{x^2 - 5x + 4}{x^2 - 2x + 1}$

42. $F(x) = \dfrac{x^2 - 2x - 15}{x^2 + 6x + 9}$

43. $G(x) = \dfrac{x}{(x + 2)^2}$

44. $G(x) = \dfrac{2 - x}{(x - 1)^2}$

45. $f(x) = x + \dfrac{1}{x}$

46. $f(x) = 2x + \dfrac{9}{x}$

47. $f(x) = x^2 + \dfrac{1}{x}$

48. $f(x) = 2x^2 + \dfrac{16}{x}$

49. $f(x) = x + \dfrac{1}{x^3}$

50. $f(x) = 2x + \dfrac{9}{x^3}$

In Problems 51–56, graph each function using a graphing utility; then use MINIMUM to obtain the minimum value, rounded to tw decimal places.

51. $f(x) = x + \dfrac{1}{x}, \quad x > 0$

52. $f(x) = 2x + \dfrac{9}{x}, \quad x > 0$

53. $f(x) = x^2 + \dfrac{1}{x}, \quad x > 0$

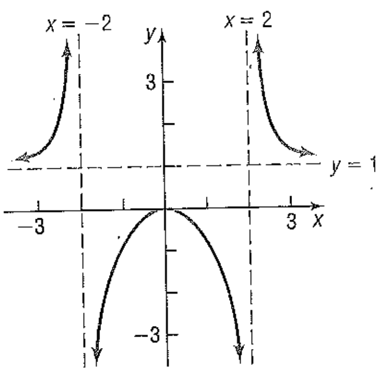
54. $f(x) = 2x^2 + \dfrac{9}{x}, \quad x > 0$

55. $f(x) = x + \dfrac{1}{x^3}, \quad x > 0$

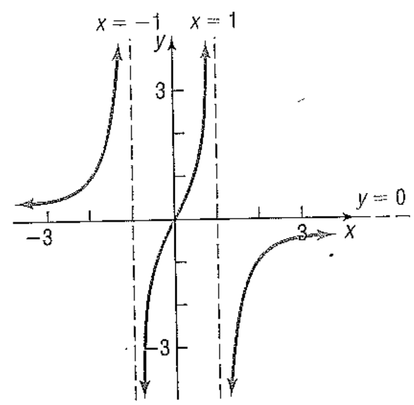
56. $f(x) = 2x + \dfrac{9}{x^3}, \quad x > 0$

In Problems 57–60, find a rational function that might have the given graph. (More than one answer might be possible.)

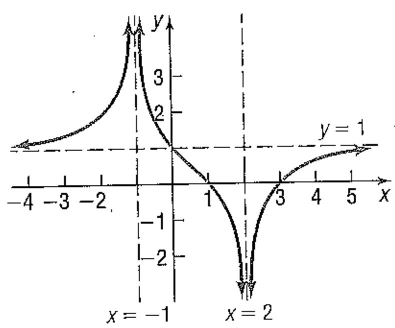
57.



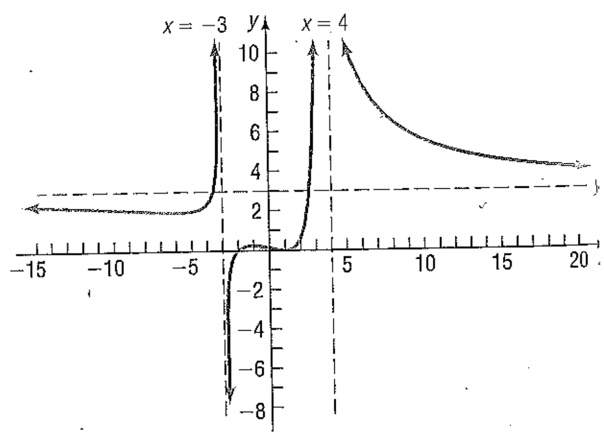
58.



59.



60.



Applications and Extensions

61. Drug Concentration The concentration C of a certain drug in a patient's bloodstream t hours after injection is given by

$$C(t) = \frac{t}{2t^2 + 1}$$

(a) Find the horizontal asymptote of $C(t)$. What happens to the concentration of the drug as t increases?

(b) Using your graphing utility, graph $C = C(t)$.

(c) Determine the time at which the concentration is highest.

62. Drug Concentration The concentration C of a certain in a patient's bloodstream t minutes after injection is give

$$C(t) = \frac{50t}{t^2 + 25}$$

(a) Find the horizontal asymptote of $C(t)$. What happens t concentration of the drug as t increases?

(b) Using your graphing utility, graph $C = C(t)$.

(c) Determine the time at which the concentrati highest.

63. Minimum Cost A rectangular area adjacent to a river is to be fenced in; no fence is needed on the river side. The enclosed area is to be 1000 square feet. Fencing for the side parallel to the river is \$5 per linear foot, and fencing for the other two sides is \$8 per linear foot; the four corner posts are \$25 apiece. Let x be the length of one of the sides perpendicular to the river.

(a) Write a function $C(x)$ that describes the cost of the project.
(b) What is the domain of C?
(c) Use a graphing utility to graph $C = C(x)$.
(d) Find the dimensions of the cheapest enclosure.

Source: http://dl.uncw.edu/digilib/mathematics/algebra/mat111hb/pandr/rational/rational.html

64. Doppler Effect The Doppler effect (named after Christian Doppler) is the change in the pitch (frequency) of the sound from a source (s) as heard by an observer (o) when one or both are in motion. If we assume both the source and the observer are moving in the same direction, the relationship is

$$f' = f_a\left(\frac{v - v_o}{v - v_s}\right)$$

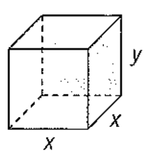
where
f' = perceived pitch by the observer
f_a = actual pitch of the source
v = speed of sound in air (assume 772.4 mph)
v_o = speed of the observer
v_s = speed of the source

Suppose that you are traveling down the road at 45 mph and you hear an ambulance (with siren) coming toward you from the rear. The actual pitch of the siren is 600 hertz (Hz).

(a) Write a function $f'(v_s)$ that describes this scenario.
(b) If $f' = 620$ Hz, find the speed of the ambulance.
(c) Use a graphing utility to graph the function.
(d) Verify your answer from part (b).

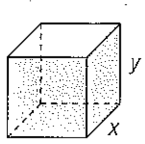
Source: www.kettering.edu/~drussell/

65. Minimizing Surface Area United Parcel Service has contracted you to design a closed box with a square base that has a volume of 10,000 cubic inches. See the illustration.



(a) Express the surface area S of the box as a function of x.
(b) Using a graphing utility, graph the function found in part (a).
(c) What is the minimum amount of cardboard that can be used to construct the box?
(d) What are the dimensions of the box that minimize the surface area?
(e) Why might UPS be interested in designing a box that minimizes the surface area?

66. Minimizing Surface Area United Parcel Service has contracted you to design an open box with a square base that has a volume of 5000 cubic inches. See the illustration.

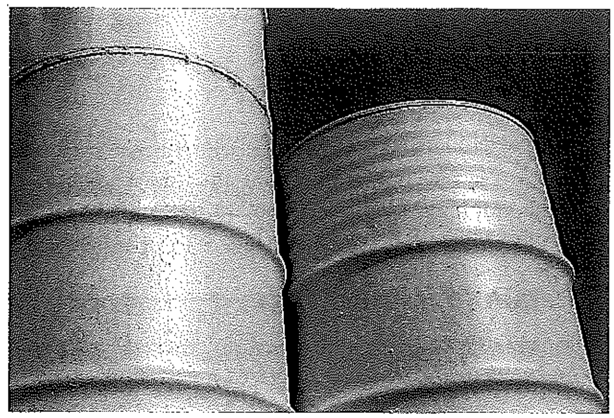


(a) Express the surface area S of the box as a function of x.
(b) Using a graphing utility, graph the function found in part (a).
(c) What is the minimum amount of cardboard that can be used to construct the box?
(d) What are the dimensions of the box that minimize the surface area?
(e) Why might UPS be interested in designing a box that minimizes the surface area?

67. Cost of a Can A can in the shape of a right circular cylinder is required to have a volume of 500 cubic centimeters. The top and bottom are made of material that costs 6¢ per square centimeter, while the sides are made of material that costs 4¢ per square centimeter.

(a) Express the total cost C of the material as a function of the radius r of the cylinder. (Refer to Figure 44.)
(b) Graph $C = C(r)$. For what value of r is the cost C a minimum?

68. Material Needed to Make a Drum A steel drum in the shape of a right circular cylinder is required to have a volume of 100 cubic feet.



(a) Express the amount A of material required to make the drum as a function of the radius r of the cylinder.
(b) How much material is required if the drum's radius is 3 feet?
(c) How much material is required if the drum's radius is 4 feet?
(d) How much material is required if the drum's radius is 5 feet?
(e) Graph $A = A(r)$. For what value of r is A smallest?

69. Removing a Discontinuity In Example 5, we analyzed the rational function $R(x) = \dfrac{2x^2 - 5x + 2}{x^2 - 4}$. We found that the graph of the rational function has a hole at the point $\left(2, \dfrac{3}{4}\right)$. Therefore, the graph of R is discontinuous at $\left(2, \dfrac{3}{4}\right)$. We could remove this discontinuity by defining the rational function R using the following piecewise-defined function:

$$R(x) = \begin{cases} \dfrac{2x^2 - 5x + 2}{x^2 - 4} & \text{if } x \neq 2 \\ \dfrac{3}{4} & \text{if } x = 2 \end{cases}$$

(a) Redefine R from Problem 33 so that the discontinuity is removed.

(b) Redefine R from Problem 35 so that the discontinuity is removed.

70. Removing a Discontinuity See Problem 69.

(a) Redefine R from Problem 34 so that the discontinuity removed.

(b) Redefine R from Problem 36 so that the discontinuity removed.

Discussion and Writing

71. Graph each of the following functions:

$$y = \frac{x^2 - 1}{x - 1} \qquad y = \frac{x^3 - 1}{x - 1}$$
$$y = \frac{x^4 - 1}{x - 1} \qquad y = \frac{x^5 - 1}{x - 1}$$

Is $x = 1$ a vertical asymptote? Why not? What is happening for $x = 1$? What do you conjecture about $y = \frac{x^n - 1}{x - 1}$, $n \geq 1$ an integer, for $x = 1$?

72. Graph each of the following functions:

$$y = \frac{x^2}{x - 1} \quad y = \frac{x^4}{x - 1} \quad y = \frac{x^6}{x - 1} \quad y = \frac{x^8}{x - 1}$$

What similarities do you see? What differences?

73. Write a few paragraphs that provide a general strategy for graphing a rational function. Be sure to mention the following: proper, improper, intercepts, and asymptotes.

74. Create a rational function that has the followi characteristics: crosses the x-axis at 2; touches the x-a at -1; one vertical asymptote at $x = -5$ and another $x = 6$; and one horizontal asymptote, $y = 3$. Compare yo function to a fellow classmate's. How do they differ? Wh are their similarities?

75. Create a rational function that has the following characteristi crosses the x-axis at 3; touches the x-axis at -2; one verti asymptote, $x = 1$; and one horizontal asymptote, $y =$ Give your rational function to a fellow classmate and ask a written critique of your rational function.

76. Create a rational function with the following characterist three real zeros, one of multiplicity 2; y-intercept, 1; verti asymptotes, $x = -2$ and $x = 3$; oblique asympt $y = 2x + 1$. Is this rational function unique? Compare y function with those of other students. What will be the sa as everyone else's? Add some more characteristics, such symmetry or naming the real zeros. How does this mo the rational function?

77. Explain the circumstances under which the graph o rational function will have a hole.

Retain Your Knowledge

Problems 78–81 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in yo mind so that you are better prepared for the final exam.

78. Subtract: $(4x^3 - 7x + 1) - (5x^2 - 9x + 3)$

79. Solve: $\frac{3x}{3x + 1} = \frac{x - 2}{x + 5}$

80. Find the maximum value of $f(x) = -\frac{2}{3}x^2 + 6x - 5$.

81. Approximate $\frac{\sqrt{5} - 3}{\sqrt{7} + 2}$. Round your answer to three decimal places.

'Are You Prepared?' Answers

1. $\left(0, \frac{1}{4}\right), (1, 0), (-1, 0)$

2. $\left\{-1, \frac{1}{2}\right\}$

3.6 Polynomial and Rational Inequalities

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solving Linear Inequalities (Appendix A, Section A.10, pp. A84–A86)
- Solving Quadratic Inequalities (Section 2.5, pp. 160–162)

Now Work the 'Are You Prepared?' problems on page 263.

OBJECTIVES 1 Solve Polynomial Inequalities (p. 259)
2 Solve Rational Inequalities (p. 260)