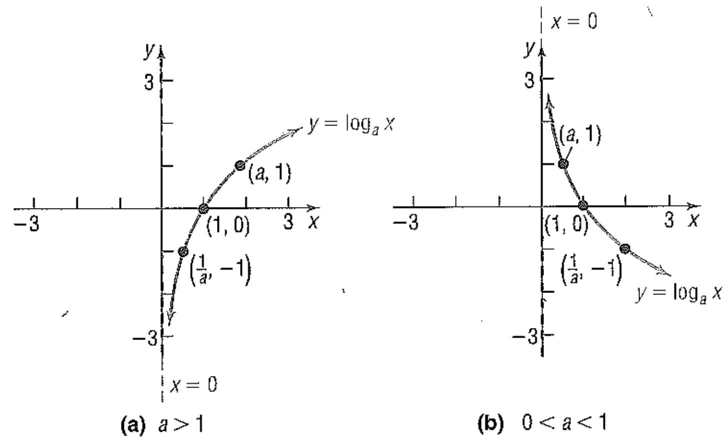
SUMMARY

Properties of the Logarithmic Function

$f(x) = \log_a x, \quad a > 1$ ($y = \log_a x$ means $x = a^y$)	Domain: the interval $(0, \infty)$; Range: the interval $(-\infty, \infty)$ x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); increasing; one-to-one See Figure 41(a) for a typical graph.
$f(x) = \log_a x, \quad 0 < a < 1$ ($y = \log_a x$ means $x = a^y$)	Domain: the interval $(0, \infty)$; Range: the interval $(-\infty, \infty)$ x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); decreasing; one-to-one See Figure 41(b) for a typical graph.

Figure 41



4.4 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Solve each inequality:
   (a) $3x - 7 \le 8 - 2x$ (pp. A84–A85)
   (b) $x^2 - x - 6 > 0$ (pp. 160–162)
2. Solve the inequality: $\dfrac{x-1}{x+4} > 0$ (pp. 260–262)
3. Solve: $2x + 3 = 9$ (pp. A64–A66)

Concepts and Vocabulary

4. The domain of the logarithmic function $f(x) = \log_a x$ is ________.
5. The graph of every logarithmic function $f(x) = \log_a x$, where $a > 0$, and $a \neq 1$, passes through three points: ______, ______, and ______.
6. If the graph of a logarithmic function $f(x) = \log_a x$, where $a > 0$ and $a \neq 1$, is increasing, then its base must be larger than ______.
7. ***True or False*** If $y = \log_a x$, then $y = a^x$.
8. ***True or False*** The graph of $f(x) = \log_a x$, where $a > 0$ and $a \neq 1$, has an x-intercept equal to 1 and no y-intercept.

Skill Building

In Problems 9–16, change each exponential statement to an equivalent statement involving a logarithm.

9. $9 = 3^2$
10. $16 = 4^2$
11. $a^2 = 1.6$
12. $a^3 = 2.1$
13. $2^x = 7.2$
14. $3^x = 4.6$
15. $e^x = 8$
16. $e^{2.2} = M$

In Problems 17–24, change each logarithmic statement to an equivalent statement involving an exponent.

17. $\log_2 8 = 3$
18. $\log_3\left(\dfrac{1}{9}\right) = -2$
19. $\log_a 3 = 6$
20. $\log_b 4 = 2$
21. $\log_3 2 = x$
22. $\log_2 6 = x$
23. $\ln 4 = x$
24. $\ln x = 4$

In Problems 25–36, find the exact value of each logarithm without using a calculator.

25. $\log_2 1$
26. $\log_8 8$
27. $\log_5 25$
28. $\log_3\left(\frac{1}{9}\right)$
29. $\log_{1/2} 16$
30. $\log_{1/3} 9$
31. $\log_{10}\sqrt{10}$
32. $\log_5 \sqrt[3]{25}$
33. $\log_{\sqrt{2}} 4$
34. $\log_{\sqrt{3}} 9$
35. $\ln\sqrt{e}$
36. $\ln e^3$

In Problems 37–48, find the domain of each function.

37. $f(x) = \ln(x - 3)$
38. $g(x) = \ln(x - 1)$
39. $F(x) = \log_2 x^2$
40. $H(x) = \log_5 x^3$
41. $f(x) = 3 - 2\log_4\left[\frac{x}{2} - 5\right]$
42. $g(x) = 8 + 5\ln(2x + 3)$
43. $f(x) = \ln\left(\frac{1}{x + 1}\right)$
44. $g(x) = \ln\left(\frac{1}{x - 5}\right)$
45. $g(x) = \log_5\left(\frac{x + 1}{x}\right)$
46. $h(x) = \log_3\left(\frac{x}{x - 1}\right)$
47. $f(x) = \sqrt{\ln x}$
48. $g(x) = \frac{1}{\ln x}$

In Problems 49–56, use a calculator to evaluate each expression. Round your answer to three decimal places.

49. $\ln\frac{5}{3}$
50. $\frac{\ln 5}{3}$
51. $\frac{\ln\frac{10}{3}}{0.04}$
52. $\frac{\ln\frac{2}{3}}{-0.1}$
53. $\frac{\ln 4 + \ln 2}{\log 4 + \log 2}$
54. $\frac{\log 15 + \log 20}{\ln 15 + \ln 20}$
55. $\frac{2\ln 5 + \log 50}{\log 4 - \ln 2}$
56. $\frac{3\log 80 - \ln 5}{\log 5 + \ln 20}$

57. Find a so that the graph of $f(x) = \log_a x$ contains the point $(2, 2)$.

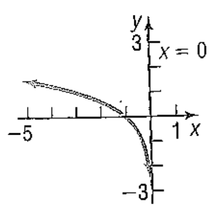
58. Find a so that the graph of $f(x) = \log_a x$ contains the point $\left(\frac{1}{2}, -4\right)$.

In Problems 59–62, graph each function and its inverse on the same Cartesian plane.

59. $f(x) = 3^x; f^{-1}(x) = \log_3 x$
60. $f(x) = 4^x; f^{-1}(x) = \log_4 x$
61. $f(x) = \left(\frac{1}{2}\right)^x; f^{-1}(x) = \log_{\frac{1}{2}} x$
62. $f(x) = \left(\frac{1}{3}\right)^x; f^{-1}(x) = \log_{\frac{1}{3}} x$

In Problems 63–70, the graph of a logarithmic function is given. Match each graph to one of the following functions.

(A) $y = \log_3 x$
(B) $y = \log_3(-x)$
(C) $y = -\log_3 x$
(D) $y = -\log_3(-x)$
(E) $y = \log_3 x - 1$
(F) $y = \log_3(x - 1)$
(G) $y = \log_3(1 - x)$
(H) $y = 1 - \log_3 x$

63.


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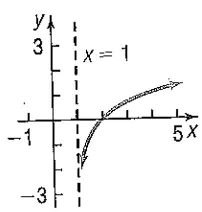

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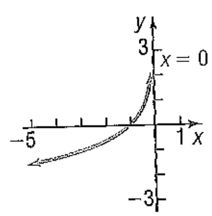

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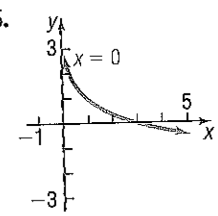

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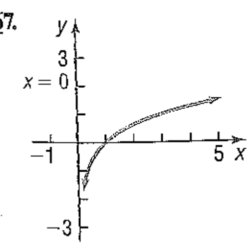

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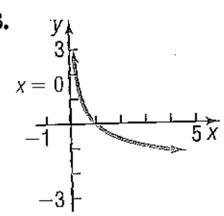

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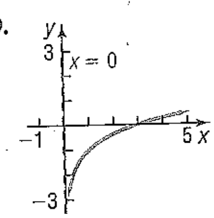

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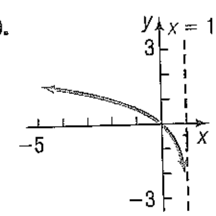

In Problems 71–86, use the given function f to:

(a) Find the domain of f. *(b) Graph f.* *(c) From the graph, determine the range and any asymptotes of f.*
(d) Find f^{-1}, the inverse of f. *(e) Find the domain and the range of f^{-1}.* *(f) Graph f^{-1}.*

71. $f(x) = \ln(x + 4)$
72. $f(x) = \ln(x - 3)$
73. $f(x) = 2 + \ln x$
74. $f(x) = -\ln(-x)$
75. $f(x) = \ln(2x) - 3$
76. $f(x) = -2\ln(x + 1)$
77. $f(x) = \log(x - 4) + 2$
78. $f(x) = \frac{1}{2}\log x - 5$

79. $f(x) = \frac{1}{2}\log(2x)$
80. $f(x) = \log(-2x)$
81. $f(x) = 3 + \log_3(x + 2)$
82. $f(x) = 2 - \log_3(x + 1)$
83. $f(x) = e^{x+2} - 3$
84. $f(x) = 3e^x + 2$
85. $f(x) = 2^{x/3} + 4$
86. $f(x) = -3^{x+1}$

In Problems 87–110, solve each equation.

87. $\log_3 x = 2$
88. $\log_5 x = 3$
89. $\log_2(2x + 1) = 3$
90. $\log_3(3x - 2) = 2$
91. $\log_x 4 = 2$
92. $\log_x\left(\frac{1}{8}\right) = 3$
93. $\ln e^x = 5$
94. $\ln e^{-2x} = 8$
95. $\log_4 64 = x$
96. $\log_5 625 = x$
97. $\log_3 243 = 2x + 1$
98. $\log_6 36 = 5x + 3$
99. $e^{3x} = 10$
100. $e^{-2x} = \frac{1}{3}$
101. $e^{2x+5} = 8$
102. $e^{-2x+1} = 13$
103. $\log_3(x^2 + 1) = 2$
104. $\log_5(x^2 + x + 4) = 2$
105. $\log_2 8^x = -3$
106. $\log_3 3^x = -1$
107. $5e^{0.2x} = 7$
108. $8 \cdot 10^{2x-7} = 3$
109. $2 \cdot 10^{2-x} = 5$
110. $4e^{x+1} = 5$

Mixed Practice

111. Suppose that $G(x) = \log_3(2x + 1) - 2$.
   (a) What is the domain of G?
   (b) What is $G(40)$? What point is on the graph of G?
   (c) If $G(x) = 3$, what is x? What point is on the graph of G?
   (d) What is the zero of G?

112. Suppose that $F(x) = \log_2(x + 1) - 3$.
   (a) What is the domain of F?
   (b) What is $F(7)$? What point is on the graph of F?
   (c) If $F(x) = -1$, what is x? What point is on the graph of F?
   (d) What is the zero of F?

In Problems 113–116, graph each function. Based on the graph, state the domain and the range, and find any intercepts.

113. $f(x) = \begin{cases} \ln(-x) & \text{if } x < 0 \\ \ln x & \text{if } x > 0 \end{cases}$

114. $f(x) = \begin{cases} \ln(-x) & \text{if } x \le -1 \\ -\ln(-x) & \text{if } -1 < x < 0 \end{cases}$

115. $f(x) = \begin{cases} -\ln x & \text{if } 0 < x < 1 \\ \ln x & \text{if } x \ge 1 \end{cases}$

116. $f(x) = \begin{cases} \ln x & \text{if } 0 < x < 1 \\ -\ln x & \text{if } x \ge 1 \end{cases}$

Applications and Extensions

117. **Chemistry** The pH of a chemical solution is given by the formula

$$\text{pH} = -\log_{10}[\text{H}^+]$$

where $[\text{H}^+]$ is the concentration of hydrogen ions in moles per liter. Values of pH range from 0 (acidic) to 14 (alkaline).
   (a) What is the pH of a solution for which $[\text{H}^+]$ is 0.1?
   (b) What is the pH of a solution for which $[\text{H}^+]$ is 0.01?
   (c) What is the pH of a solution for which $[\text{H}^+]$ is 0.001?
   (d) What happens to pH as the hydrogen ion concentration decreases?
   (e) Determine the hydrogen ion concentration of an orange (pH = 3.5).
   (f) Determine the hydrogen ion concentration of human blood (pH = 7.4).

118. **Diversity Index Shannon's diversity index** is a measure of the diversity of a population. The diversity index is given by the formula

$$H = -(p_1 \log p_1 + p_2 \log p_2 + \cdots + p_n \log p_n)$$

where p_1 is the proportion of the population that is species 1, p_2 is the proportion of the population that is species 2, and so on.

   (a) According to the U.S. Census Bureau, the distribution of race in the United States in 2010 was as follows:

Race	Proportion
White	0.724
Black or African American	0.126
American Indian and Alaska Native	0.009
Asian	0.048
Native Hawaiian and Other Pacific Islander	0.002
Some Other Race	0.062
Two or More Races	0.029

Source: U.S. Census Bureau

   Compute the diversity index of the United States in 2010.
   (b) The largest value of the diversity index is given by $H_{\max} = \log(S)$, where S is the number of categories of race. Compute $H_{\max}$.
   (c) The **evenness ratio** is given by $E_H = \frac{H}{H_{\max}}$, where $0 \le E_H \le 1$. If $E_H = 1$, there is complete evenness. Compute the evenness ratio for the United States.

(d) Obtain the distribution of race for the United States in 2000 from the Census Bureau. Compute Shannon's diversity index. Is the United States becoming more diverse? Why?

119. Atmospheric Pressure The atmospheric pressure p on an object decreases with increasing height. This pressure, measured in millimeters of mercury, is related to the height h (in kilometers) above sea level by the function

$$p(h) = 760e^{-0.145h}$$

(a) Find the height of an aircraft if the atmospheric pressure is 320 millimeters of mercury.
(b) Find the height of a mountain if the atmospheric pressure is 667 millimeters of mercury.

120. Healing of Wounds The normal healing of wounds can be modeled by an exponential function. If A_0 represents the original area of the wound, and if A equals the area of the wound, then the function

$$A(n) = A_0e^{-0.35n}$$

describes the area of a wound after n days following an injury when no infection is present to retard the healing. Suppose that a wound initially had an area of 100 square millimeters.
(a) If healing is taking place, after how many days will the wound be one-half its original size?
(b) How long before the wound is 10% of its original size?

121. Exponential Probability Between 12:00 PM and 1:00 PM, cars arrive at Citibank's drive-thru at the rate of 6 cars per hour (0.1 car per minute). The following formula from statistics can be used to determine the probability that a car will arrive within t minutes of 12:00 PM.

$$F(t) = 1 - e^{-0.1t}$$

(a) Determine how many minutes are needed for the probability to reach 50%.
(b) Determine how many minutes are needed for the probability to reach 80%.
(c) Is it possible for the probability to equal 100%? Explain.

122. Exponential Probability Between 5:00 PM and 6:00 PM, cars arrive at Jiffy Lube at the rate of 9 cars per hour (0.15 car per minute). The following formula from statistics can be used to determine the probability that a car will arrive within t minutes of 5:00 PM.

$$F(t) = 1 - e^{-0.15t}$$

(a) Determine how many minutes are needed for the probability to reach 50%.
(b) Determine how many minutes are needed for the probability to reach 80%.

123. Drug Medication The formula

$$D = 5e^{-0.4h}$$

can be used to find the number of milligrams D of a certain drug that is in a patient's bloodstream h hours after the drug was administered. When the number of milligrams reaches 2, the drug is to be administered again. What is the time between injections?

124. Spreading of Rumors A model for the number N of people in a college community who have heard a certain rumor is

$$N = P(1 - e^{-0.15d})$$

where P is the total population of the community and d is the number of days that have elapsed since the rumor began. In a community of 1000 students, how many days will elapse before 450 students have heard the rumor?

125. Current in a *RL* Circuit The equation governing the amount of current I (in amperes) after time t (in seconds) in a simple RL circuit consisting of a resistance R (in ohms), an inductance L (in henrys), and an electromotive force E (in volts) is

$$I = \frac{E}{R}[1 - e^{-(R/L)t}]$$

If $E = 12$ volts, $R = 10$ ohms, and $L = 5$ henrys, how long does it take to obtain a current of 0.5 ampere? Of 1.0 ampere? Graph the equation.

126. Learning Curve Psychologists sometimes use the function

$$L(t) = A(1 - e^{-kt})$$

to measure the amount L learned at time t. The number A represents the amount to be learned, and the number k measures the rate of learning. Suppose that a student has an amount A of 200 vocabulary words to learn. A psychologist determines that the student had learned 20 vocabulary words after 5 minutes.
(a) Determine the rate of learning k.
(b) Approximately how many words will the student have learned after 10 minutes?
(c) After 15 minutes?
(d) How long does it take for the student to learn 180 words?

Loudness of Sound Problems 127–130 use the following discussion: The ***loudness*** *$L(x)$, measured in decibels (dB), of a sound of intensity x, measured in watts per square meter, is defined as $L(x) = 10\log\frac{x}{I_0}$, where $I_0 = 10^{-12}$ watt per square meter is the least intense sound that a human ear can detect. Determine the loudness, in decibels, of each of the following sounds.*

127. Normal conversation: intensity of $x = 10^{-7}$ watt per square meter.

128. Amplified rock music: intensity of 10^{-1} watt per square meter.

129. Heavy city traffic: intensity of $x = 10^{-3}$ watt per square meter.

130. Diesel truck traveling 40 miles per hour 50 feet away: intensity 10 times that of a passenger car traveling 50 miles per hour 50 feet away whose loudness is 70 decibels.

The Richter Scale Problems 131 and 132 use the following discussion: The ***Richter scale*** *is one way of converting seismographic readings into numbers that provide an easy reference for measuring the magnitude M of an earthquake. All earthquakes are compared to a* ***zero-level earthquake*** *whose seismographic reading measures 0.001 millimeter at a distance of 100 kilometers from the epicenter. An earthquake whose seismographic reading measures x millimeters has* ***magnitude*** *$M(x)$, given by*

$$M(x) = \log\left(\frac{x}{x_0}\right)$$

where $x_0 = 10^{-3}$ is the reading of a zero-level earthquake the same distance from its epicenter. In Problems 131 and 132, determine the magnitude of each earthquake.

131. Magnitude of an Earthquake Mexico City in 1985: seismographic reading of 125,892 millimeters 100 kilometers from the center

132. Magnitude of an Earthquake San Francisco in 1906: seismographic reading of 50,119 millimeters 100 kilometers from the center

133. Alcohol and Driving The concentration of alcohol in a person's bloodstream is measurable. Suppose that the relative risk R of having an accident while driving a car can be modeled by an equation of the form

$$R = e^{kx}$$

where x is the percent of concentration of alcohol in the bloodstream and k is a constant.

(a) Suppose that a concentration of alcohol in the bloodstream of 0.03 percent results in a relative risk of an accident of 1.4. Find the constant k in the equation.

(b) Using this value of k, what is the relative risk if the concentration is 0.17 percent?

(c) Using the same value of k, what concentration of alcohol corresponds to a relative risk of 100?

(d) If the law asserts that anyone with a relative risk of having an accident of 5 or more should not have driving privileges, at what concentration of alcohol in the bloodstream should a driver be arrested and charged with DUI?

(e) Compare this situation with that of Example 10. If you were a lawmaker, which situation would you support? Give your reasons.

Discussion and Writing

134. Is there any function of the form $y = x^\alpha$, $0 < \alpha < 1$, that increases more slowly than a logarithmic function whose base is greater than 1? Explain.

135. In the definition of the logarithmic function, the base a is not allowed to equal 1. Why?

136. **Critical Thinking** In buying a new car, one consideration might be how well the price of the car holds up over time. Different makes of cars have different depreciation rates. One way to compute a depreciation rate for a car is given here. Suppose that the current prices of a certain automobile are as shown in the table.

Age in Years					
New	1	2	3	4	5
\$38,000	\$36,600	\$32,400	\$28,750	\$25,400	\$21,200

Use the formula New = Old(e^{Rt}) to find R, the annual depreciation rate, for a specific time t. When might be the best time to trade in the car? Consult the NADA ("blue") book and compare two like models that you are interested in. Which has the better depreciation rate?

Retain Your Knowledge

Problems 137–140 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

137. Find the real zeros of $g(x) = 4x^4 - 37x^2 + 9$. What are the x-intercepts of the graph of g?

138. Find the average rate of change of $f(x) = 9^x$ from $\frac{1}{2}$ to 1.

139. Use the Intermediate Value Theorem to show that the function $f(x) = 4x^3 - 2x^2 - 7$ has a real zero in the interval $[1, 2]$.

140. A complex polynomial function f of degree 4 has the zeros -1, 2, and $3 - i$. Find the remaining zero(s) of f. Then find a polynomial function with real coefficients that has the zeros.

'Are You Prepared?' Answers

1. (a) $\{x \mid x \le 3\}$ (b) $\{x \mid x < -2 \text{ or } x > 3\}$ **2.** $\{x \mid x < -4 \text{ or } x > 1\}$ **3.** $\{3\}$

4.5 Properties of Logarithms

OBJECTIVES
1 Work with the Properties of Logarithms (p. 324)
2 Write a Logarithmic Expression as a Sum or Difference of Logarithms (p. 327)
3 Write a Logarithmic Expression as a Single Logarithm (p. 327)
4 Evaluate a Logarithm Whose Base Is Neither 10 Nor e (p. 328)
5 Graph a Logarithmic Function Whose Base Is Neither 10 Nor e (p. 330)

1 Work with the Properties of Logarithms

Logarithms have some very useful properties that can be derived directly from the definition and the laws of exponents.