Solution Let $\theta = \tan^{-1} u$ so that $\tan\theta = u$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $-\infty < u < \infty$. This means that $\sec\theta > 0$. Then

$$\sin(\tan^{-1} u) = \sin\theta = \sin\theta \cdot \frac{\cos\theta}{\cos\theta} = \tan\theta\cos\theta = \frac{\tan\theta}{\sec\theta} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{u}{\sqrt{1+u^2}}$$

Multiply by 1: $\frac{\cos\theta}{\cos\theta}$; $\frac{\sin\theta}{\cos\theta} = \tan\theta$; $\sec^2\theta = 1 + \tan^2\theta$, $\sec\theta > 0$

Now Work PROBLEM 57

6.2 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. What are the domain and the range of $y = \sec x$? (pp. 407–409)
2. *True or False* The graph of $y = \sec x$ is one-to-one on the interval $\left[0, \frac{\pi}{2}\right)$ and on the interval $\left(\frac{\pi}{2}, \pi\right]$. (pp. 439–440)
3. If $\tan\theta = \frac{1}{2}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $\sin\theta =$ ________. (pp. 413–416)

Concepts and Vocabulary

4. $y = \sec^{-1} x$ means ________, where $|x|$ ______ and ______ $\le y \le$ __, $y \ne \frac{\pi}{2}$.
5. To find the inverse secant of a real number x such that $|x| \ge 1$, convert the inverse secant to an inverse ________.
6. *True or False* It is impossible to obtain exact values for the inverse secant function.
7. *True or False* $\csc^{-1} 0.5$ is not defined.
8. *True or False* The domain of the inverse cotangent function is the set of real numbers.

Skill Building

In Problems 9–36, find the exact value of each expression.

9. $\cos\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$
10. $\sin\left(\cos^{-1}\frac{1}{2}\right)$
11. $\tan\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$
12. $\tan\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$
13. $\sec\left(\cos^{-1}\frac{1}{2}\right)$
14. $\cot\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$
15. $\csc(\tan^{-1} 1)$
16. $\sec(\tan^{-1}\sqrt{3})$
17. $\sin[\tan^{-1}(-1)]$
18. $\cos\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$
19. $\sec\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$
20. $\csc\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$
21. $\cos^{-1}\left(\sin\frac{5\pi}{4}\right)$
22. $\tan^{-1}\left(\cot\frac{2\pi}{3}\right)$
23. $\sin^{-1}\left[\cos\left(-\frac{7\pi}{6}\right)\right]$
24. $\cos^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right]$
25. $\tan\left(\sin^{-1}\frac{1}{3}\right)$
26. $\tan\left(\cos^{-1}\frac{1}{3}\right)$
27. $\sec\left(\tan^{-1}\frac{1}{2}\right)$
28. $\cos\left(\sin^{-1}\frac{\sqrt{2}}{3}\right)$
29. $\cot\left[\sin^{-1}\left(-\frac{\sqrt{2}}{3}\right)\right]$
30. $\csc[\tan^{-1}(-2)]$
31. $\sin[\tan^{-1}(-3)]$
32. $\cot\left[\cos^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right]$
33. $\sec\left(\sin^{-1}\frac{2\sqrt{5}}{5}\right)$
34. $\csc\left(\tan^{-1}\frac{1}{2}\right)$
35. $\sin^{-1}\left(\cos\frac{3\pi}{4}\right)$
36. $\cos^{-1}\left(\sin\frac{7\pi}{6}\right)$

In Problems 37–44, find the exact value of each expression.

37. $\cot^{-1}\sqrt{3}$
38. $\cot^{-1} 1$
39. $\csc^{-1}(-1)$
40. $\csc^{-1}\sqrt{2}$
41. $\sec^{-1}\frac{2\sqrt{3}}{3}$
42. $\sec^{-1}(-2)$
43. $\cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)$
44. $\csc^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$

In Problems 45–56, use a calculator to find the value of each expression rounded to two decimal places.

45. $\sec^{-1} 4$
46. $\csc^{-1} 5$
47. $\cot^{-1} 2$
48. $\sec^{-1}(-3)$
49. $\csc^{-1}(-3)$
50. $\cot^{-1}\left(-\frac{1}{2}\right)$
51. $\cot^{-1}(-\sqrt{5})$
52. $\cot^{-1}(-8.1)$
53. $\csc^{-1}\left(-\frac{3}{2}\right)$
54. $\sec^{-1}\left(-\frac{4}{3}\right)$
55. $\cot^{-1}\left(-\frac{3}{2}\right)$
56. $\cot^{-1}(-\sqrt{10})$

In Problems 57–66, write each trigonometric expression as an algebraic expression in u.

57. $\cos(\tan^{-1} u)$
58. $\sin(\cos^{-1} u)$
59. $\tan(\sin^{-1} u)$
60. $\tan(\cos^{-1} u)$
61. $\sin(\sec^{-1} u)$
62. $\sin(\cot^{-1} u)$
63. $\cos(\csc^{-1} u)$
64. $\cos(\sec^{-1} u)$
65. $\tan(\cot^{-1} u)$
66. $\tan(\sec^{-1} u)$

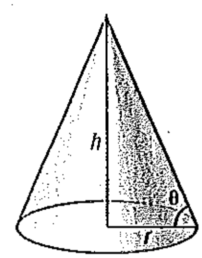
Mixed Practice

In Problems 67–78, $f(x) = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$; $g(x) = \cos x, 0 \le x \le \pi$; *and* $h(x) = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$. *Find the exact value of each composite function.*

67. $g\left(f^{-1}\left(\frac{12}{13}\right)\right)$
68. $f\left(g^{-1}\left(\frac{5}{13}\right)\right)$
69. $g^{-1}\left(f\left(\frac{7\pi}{4}\right)\right)$
70. $f^{-1}\left(g\left(\frac{5\pi}{6}\right)\right)$
71. $h\left(f^{-1}\left(-\frac{3}{5}\right)\right)$
72. $h\left(g^{-1}\left(-\frac{4}{5}\right)\right)$
73. $g\left(h^{-1}\left(\frac{12}{5}\right)\right)$
74. $f\left(h^{-1}\left(\frac{5}{12}\right)\right)$
75. $g^{-1}\left(f\left(-\frac{4\pi}{3}\right)\right)$
76. $g^{-1}\left(f\left(-\frac{5\pi}{6}\right)\right)$
77. $h\left(g^{-1}\left(-\frac{1}{4}\right)\right)$
78. $h\left(f^{-1}\left(-\frac{2}{5}\right)\right)$

Applications and Extensions

Problems 79 and 80 require the following discussion: When granular materials are allowed to fall freely, they form conical (cone-shaped) piles. The naturally occurring angle of slope, measured from the horizontal, at which the loose material comes to rest is called the ***angle of repose*** *and varies for different materials. The angle of repose* θ *is related to the height h and base radius r of the conical pile by the equation* $\theta = \cot^{-1}\frac{r}{h}$. *See the illustration.*



79. **Angle of Repose: De-icing Salt** Due to potential transportation issues (for example, frozen waterways), de-icing salt used by highway departments in the Midwest must be ordered early and stored for future use. When de-icing salt is stored in a pile 14 feet high, the diameter of the base of the pile is 45 feet.
   (a) Find the angle of repose for de-icing salt.
   (b) What is the base diameter of a pile that is 17 feet high?
   (c) What is the height of a pile that has a base diameter of approximately 122 feet?
   Source: Salt Institute, *The Salt Storage Handbook,* 2006

80. **Angle of Repose: Bunker Sand** The steepness of sand bunkers on a golf course is affected by the angle of repose of the sand (a larger angle of repose allows for steeper bunkers). A freestanding pile of loose sand from a United States Golf Association (USGA) bunker had a height of 4 feet and a base diameter of approximately 6.68 feet.
   (a) Find the angle of repose for USGA bunker sand.
   (b) What is the height of such a pile if the diameter of the base is 8 feet?
   (c) A 6-foot-high pile of loose Tour Grade 50/50 sand has a base diameter of approximately 8.44 feet. Which type of sand (USGA or Tour Grade 50/50) would be better suited for steep bunkers?
   Source: 2004 Annual Report, Purdue University Turfgrass Science Program

81. **Artillery** A projectile fired into the first quadrant from the origin of a coordinate system will pass through the point (x, y) at time t according to the relationship $\cot\theta = \frac{2x}{2y + gt^2}$, where θ = the angle of elevation of the launcher and g = the acceleration due to gravity = 32.2 feet/second^2. An artilleryman is firing at an enemy bunker located 2450 feet up the side of a hill that is 6175 feet away. He fires a round, and exactly 2.27 seconds later he scores a direct hit.
   (a) What angle of elevation did he use?
   (b) If the angle of elevation is also given by $\sec\theta = \frac{v_0 t}{x}$, where v_0 is the muzzle velocity of the weapon, find the muzzle velocity of the artillery piece he used.
   Source: www.egwald.com/geometry/projectile3d.php

82. Using a graphing utility, graph $y = \cot^{-1}x$.
83. Using a graphing utility, graph $y = \sec^{-1}x$.
84. Using a graphing utility, graph $y = \csc^{-1}x$.

Discussion and Writing

85. Explain in your own words how you would use your calculator to find the value of $\cot^{-1} 10$.

86. Consult three books on calculus, and then write down the definition in each of $y = \sec^{-1} x$ and $y = \csc^{-1} x$. Compare these with the definitions given in this text.

Retain Your Knowledge

Problems 87–90 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

87. Find the complex zeros of $f(x) = x^4 + 21x^2 - 100$.

88. Determine algebraically whether $f(x) = x^3 + x^2 - x$ is even, odd, or neither.

89. Convert 315° to radians.

90. Find the length of the arc subtended by a central angle of 75° on a circle of radius 6 inches. Give both the exact length and an approximation rounded to two decimal places.

'Are You Prepared?' Answers

1. Domain: $\left\{x \middle| x \neq \text{odd integer multiples of } \frac{\pi}{2}\right\}$; range: $\{y | y \leq -1 \text{ or } y \geq 1\}$ **2.** True **3.** $\frac{\sqrt{5}}{5}$

6.3 Trigonometric Equations

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Solving Equations (Appendix A, Section A.8, pp. A63–A66)
- Values of the Trigonometric Functions (Section 5.2, pp. 392–400)
- Solving Quadratic Equations (Section 2.3, pp. 137–144)
- Equations Quadratic in Form (Section 2.3, pp. 144–145)
- Using a Graphing Utility to Solve Equations (Appendix B, Section B.4, pp. B6–B7)

Now Work the 'Are You Prepared?' problems on page 496.

OBJECTIVES
1 Solve Equations Involving a Single Trigonometric Function (p. 482)
2 Solve Trigonometric Equations Using a Calculator (p. 485)
3 Solve Trigonometric Equations Quadratic in Form (p. 485)
4 Solve Trigonometric Equations Using Fundamental Identities (p. 486)
5 Solve Trigonometric Equations Using a Graphing Utility (p. 487)

1 Solve Equations Involving a Single Trigonometric Function

In this section, we discuss **trigonometric equations**—that is, equations involving trigonometric functions that are satisfied only by some values of the variable (or, possibly, are not satisfied by any values of the variable). The values that satisfy the equation are called **solutions** of the equation.

EXAMPLE 1 **Checking Whether a Given Number Is a Solution of a Trigonometric Equation**

Determine whether $\theta = \frac{\pi}{4}$ is a solution of the equation $2\sin\theta - 1 = 0$. Is $\theta = \frac{\pi}{6}$ a solution?

Solution Replace θ by $\frac{\pi}{4}$ in the given equation. The result is

$$2\sin\frac{\pi}{4} - 1 = 2\cdot\frac{\sqrt{2}}{2} - 1 = \sqrt{2} - 1 \neq 0$$

Therefore, $\frac{\pi}{4}$ is not a solution.

Next replace θ by $\frac{\pi}{6}$ in the equation. The result is

$$2\sin\frac{\pi}{6} - 1 = 2\cdot\frac{1}{2} - 1 = 0$$

Therefore, $\frac{\pi}{6}$ is a solution of the given equation.