**EXAMPLE 9** **Solving a Trigonometric Equation Using Identities**

Solve the equation: $\cos^2\theta + \sin\theta = 2, \quad 0 \le \theta < 2\pi$

**Solution** This equation involves two trigonometric functions, sine and cosine. Using a form of the Pythagorean Identity, $\sin^2\theta + \cos^2\theta = 1$, rewrite the equation in terms of $\sin\theta$.

$$\cos^2\theta + \sin\theta = 2$$
$$(1 - \sin^2\theta) + \sin\theta = 2 \quad \cos^2\theta = 1 - \sin^2\theta$$
$$\sin^2\theta - \sin\theta + 1 = 0$$

This is a quadratic equation in $\sin\theta$. The discriminant is $b^2 - 4ac = 1 - 4 = -3 < 0$. Therefore, the equation has no real solution. The solution set is the empty set, $\varnothing$.

✓Check: Graph $Y_1 = \cos^2 x + \sin x$ and $Y_2 = 2$ to see that the two graphs never intersect, so the equation $Y_1 = Y_2$ has no real solution.

## 5 Solve Trigonometric Equations Using a Graphing Utility

The techniques introduced in this section apply only to certain types of trigonometric equations. Solutions for other types are usually studied in calculus, using numerical methods.

**EXAMPLE 10** **Solving a Trigonometric Equation Using a Graphing Utility**

Solve: $5 \sin x + x = 3$

Express the solution(s) rounded to two decimal places.

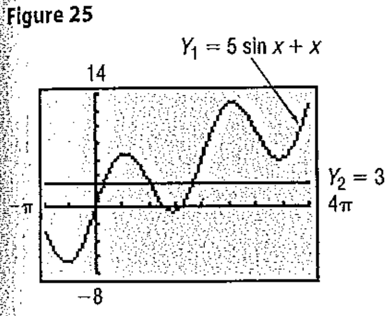
**Solution** This type of trigonometric equation cannot be solved by previous methods. A graphing utility, though, can be used here. Each solution of this equation is the $x$-coordinate of a point of intersection of the graphs of $Y_1 = 5\sin x + x$ and $Y_2 = 3$. See Figure 25.

There are three points of intersection; the $x$-coordinates provide the solutions. Use INTERSECT to find

$$x = 0.52, \qquad x = 3.18, \qquad x = 5.71$$

The solution set is $\{0.52, 3.18, 5.71\}$.

Figure 25




Now Work PROBLEM 81

# 6.3 Assess Your Understanding

**'Are You Prepared?'** *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Solve: $3x - 5 = -x + 1$ (pp. A63–A65)
2. $\sin\left(\frac{\pi}{4}\right) =$ ___; $\cos\left(\frac{8\pi}{3}\right) =$ ___. (pp. 395–400)
3. Find the real solutions of $4x^2 - x - 5 = 0$. (pp. A67–A68)
4. Find the real solutions of $x^2 - x - 1 = 0$. (pp. 140–143)
5. Find the real solutions of $(2x - 1)^2 - 3(2x - 1) - 4 = 0$. (pp. 144–145)
6. Use a graphing utility to solve $5x^3 - 2 = x - x^2$. Round answers to two decimal places. (pp. B6–B7)

## Concepts and Vocabulary

7. Two solutions of the equation $\sin\theta = \frac{1}{2}$ are __ and ___.
8. All the solutions of the equation $\sin\theta = \frac{1}{2}$ are ___.
9. *True or False* Most trigonometric equations have unique solutions.
10. *True or False* The equation $\sin\theta = 2$ has a real solution that can be found using a calculator.

## Skill Building

*In Problems 11–34, solve each equation on the interval* $0 \le \theta < 2\pi$.

11. $2\sin\theta + 3 = 2$
12. $1 - \cos\theta = \frac{1}{2}$
13. $4\cos^2\theta = 1$

14. $\tan^2\theta = \frac{1}{3}$

15. $2\sin^2\theta - 1 = 0$

16. $4\cos^2\theta - 3 = 0$

17. $\sin(3\theta) = -1$

18. $\tan\frac{\theta}{2} = \sqrt{3}$

19. $\cos(2\theta) = -\frac{1}{2}$

20. $\tan(2\theta) = -1$

21. $\sec\frac{3\theta}{2} = -2$

22. $\cot\frac{2\theta}{3} = -\sqrt{3}$

23. $2\sin\theta + 1 = 0$

24. $\cos\theta + 1 = 0$

25. $\tan\theta + 1 = 0$

26. $\sqrt{3}\cot\theta + 1 = 0$

27. $4\sec\theta + 6 = -2$

28. $5\csc\theta - 3 = 2$

29. $3\sqrt{2}\cos\theta + 2 = -1$

30. $4\sin\theta + 3\sqrt{3} = \sqrt{3}$

31. $\cos\left(2\theta - \frac{\pi}{2}\right) = -1$

32. $\sin\left(3\theta + \frac{\pi}{18}\right) = 1$

33. $\tan\left(\frac{\theta}{2} + \frac{\pi}{3}\right) = 1$

34. $\cos\left(\frac{\theta}{3} - \frac{\pi}{4}\right) = \frac{1}{2}$

*In Problems 35–44, solve each equation. Give a general formula for all the solutions. List six solutions.*

35. $\sin\theta = \frac{1}{2}$

36. $\tan\theta = 1$

37. $\tan\theta = -\frac{\sqrt{3}}{3}$

38. $\cos\theta = -\frac{\sqrt{3}}{2}$

39. $\cos\theta = 0$

40. $\sin\theta = \frac{\sqrt{2}}{2}$

41. $\cos(2\theta) = -\frac{1}{2}$

42. $\sin(2\theta) = -1$

43. $\sin\frac{\theta}{2} = -\frac{\sqrt{3}}{2}$

44. $\tan\frac{\theta}{2} = -1$

*In Problems 45–56, use a calculator to solve each equation on the interval $0 \le \theta < 2\pi$. Round answers to two decimal places.*

45. $\sin\theta = 0.4$

46. $\cos\theta = 0.6$

47. $\tan\theta = 5$

48. $\cot\theta = 2$

49. $\cos\theta = -0.9$

50. $\sin\theta = -0.2$

51. $\sec\theta = -4$

52. $\csc\theta = -3$

53. $5\tan\theta + 9 = 0$

54. $4\cot\theta = -5$

55. $3\sin\theta - 2 = 0$

56. $4\cos\theta + 3 = 0$

*In Problems 57–80, solve each equation on the interval $0 \le \theta < 2\pi$.*

57. $2\cos^2\theta + \cos\theta = 0$

58. $\sin^2\theta - 1 = 0$

59. $2\sin^2\theta - \sin\theta - 1 = 0$

60. $2\cos^2\theta + \cos\theta - 1 = 0$

61. $(\tan\theta - 1)(\sec\theta - 1) = 0$

62. $(\cot\theta + 1)\left(\csc\theta - \frac{1}{2}\right) = 0$

63. $\sin^2\theta - \cos^2\theta = 1 + \cos\theta$

64. $\cos^2\theta - \sin^2\theta + \sin\theta = 0$

65. $\sin^2\theta = 6(\cos(-\theta) + 1)$

66. $2\sin^2\theta = 3(1 - \cos(-\theta))$

67. $\cos\theta = -\sin(-\theta)$

68. $\cos\theta - \sin(-\theta) = 0$

69. $\tan\theta = 2\sin\theta$

70. $\tan\theta = \cot\theta$

71. $1 + \sin\theta = 2\cos^2\theta$

72. $\sin^2\theta = 2\cos\theta + 2$

73. $2\sin^2\theta - 5\sin\theta + 3 = 0$

74. $2\cos^2\theta - 7\cos\theta - 4 = 0$

75. $3(1 - \cos\theta) = \sin^2\theta$

76. $4(1 + \sin\theta) = \cos^2\theta$

77. $\tan^2\theta = \frac{3}{2}\sec\theta$

78. $\csc^2\theta = \cot\theta + 1$

79. $\sec^2\theta + \tan\theta = 0$

80. $\sec\theta = \tan\theta + \cot\theta$

*In Problems 81–92, use a graphing utility to solve each equation. Express the solution(s) rounded to two decimal places.*

81. $x + 5\cos x = 0$

82. $x - 4\sin x = 0$

83. $22x - 17\sin x = 3$

84. $19x + 8\cos x = 2$

85. $\sin x + \cos x = x$

86. $\sin x - \cos x = x$

87. $x^2 - 2\cos x = 0$

88. $x^2 + 3\sin x = 0$

89. $x^2 - 2\sin(2x) = 3x$

90. $x^2 = x + 3\cos(2x)$

91. $6\sin x - e^x = 2, \quad x > 0$

92. $4\cos(3x) - e^x = 1, \quad x > 0$

## Mixed Practice

93. What are the zeros of $f(x) = 4\sin^2 x - 3$ on the interval $[0, 2\pi]$?

94. What are the zeros of $f(x) = 2\cos(3x) + 1$ on the interval $[0, \pi]$?

95. $f(x) = 3\sin x$
   (a) Find the zeros of $f$ on the interval $[-2\pi, 4\pi]$.
   (b) Graph $f(x) = 3\sin x$ on the interval $[-2\pi, 4\pi]$.
   (c) Solve $f(x) = \frac{3}{2}$ on the interval $[-2\pi, 4\pi]$. What points are on the graph of $f$? Label these points on the graph drawn in part (b).

(d) Use the graph drawn in part (b) along with the results of part (c) to determine the values of $x$ such that $f(x) > \frac{3}{2}$ on the interval $[-2\pi, 4\pi]$.

96. $f(x) = 2\cos x$

(a) Find the zeros of $f$ on the interval $[-2\pi, 4\pi]$.

(b) Graph $f(x) = 2\cos x$ on the interval $[-2\pi, 4\pi]$.

(c) Solve $f(x) = -\sqrt{3}$ on the interval $[-2\pi, 4\pi]$. What points are on the graph of $f$? Label these points on the graph drawn in part (b).

(d) Use the graph drawn in part (b) along with the results of part (c) to determine the values of $x$ such that $f(x) < -\sqrt{3}$ on the interval $[-2\pi, 4\pi]$.

97. $f(x) = 4\tan x$

(a) Solve $f(x) = -4$.

(b) For what values of $x$ is $f(x) < -4$ on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$?

98. $f(x) = \cot x$

(a) Solve $f(x) = -\sqrt{3}$.

(b) For what values of $x$ is $f(x) > -\sqrt{3}$ on the interval $(0, \pi)$?

99. (a) Graph $f(x) = 3\sin(2x) + 2$ and $g(x) = \frac{7}{2}$ on the same Cartesian plane for the interval $[0, \pi]$.

(b) Solve $f(x) = g(x)$ on the interval $[0, \pi]$, and label the points of intersection on the graph drawn in part (a).

(c) Solve $f(x) > g(x)$ on the interval $[0, \pi]$.

(d) Shade the region bounded by $f(x) = 3\sin(2x) + 2$ and $g(x) = \frac{7}{2}$ between the two points found in part (b) on the graph drawn in part (a).

100. (a) Graph $f(x) = 2\cos\frac{x}{2} + 3$ and $g(x) = 4$ on the same Cartesian plane for the interval $[0, 4\pi]$.

(b) Solve $f(x) = g(x)$ on the interval $[0, 4\pi]$, and label the points of intersection on the graph drawn in part (a).

(c) Solve $f(x) < g(x)$ on the interval $[0, 4\pi]$.

(d) Shade the region bounded by $f(x) = 2\cos\frac{x}{2} + 3$ and $g(x) = 4$ between the two points found in part (b) on the graph drawn in part (a).

101. (a) Graph $f(x) = -4\cos x$ and $g(x) = 2\cos x + 3$ on the same Cartesian plane for the interval $[0, 2\pi]$.

(b) Solve $f(x) = g(x)$ on the interval $[0, 2\pi]$, and label the points of intersection on the graph drawn in part (a).

(c) Solve $f(x) > g(x)$ on the interval $[0, 2\pi]$.

(d) Shade the region bounded by $f(x) = -4\cos x$ and $g(x) = 2\cos x + 3$ between the two points found in part (b) on the graph drawn in part (a).

102. (a) Graph $f(x) = 2\sin x$ and $g(x) = -2\sin x + 2$ on the same Cartesian plane for the interval $[0, 2\pi]$.

(b) Solve $f(x) = g(x)$ on the interval $[0, 2\pi]$, and label the points of intersection on the graph drawn in part (a).

(c) Solve $f(x) > g(x)$ on the interval $[0, 2\pi]$.

(d) Shade the region bounded by $f(x) = 2\sin x$ and $g(x) = -2\sin x + 2$ between the two points found in part (b) on the graph drawn in part (a).

## Applications and Extensions

103. **Blood Pressure** Blood pressure is a way of measuring the amount of force exerted on the walls of blood vessels. It is measured using two numbers: systolic (as the heart beats) blood pressure and diastolic (as the heart rests) blood pressure. Blood pressures vary substantially from person to person, but a typical blood pressure is 120/80, which means the systolic blood pressure is 120 mmHg and the diastolic blood pressure is 80 mmHg. Assuming that a person's heart beats 70 times per minute, the blood pressure $P$ of an individual after $t$ seconds can be modeled by the function

$$P(t) = 100 + 20\sin\left(\frac{7\pi}{3}t\right)$$

(a) In the interval [0, 1], determine the times at which the blood pressure is 100 mmHg.

(b) In the interval [0, 1], determine the times at which the blood pressure is 120 mmHg.

(c) In the interval [0, 1], determine the times at which the blood pressure is between 100 and 105 mmHg.

104. **The Ferris Wheel** In 1893, George Ferris engineered the Ferris wheel. It was 250 feet in diameter. If a Ferris wheel makes 1 revolution every 40 seconds, then the function

$$h(t) = 125\sin\left(0.157t - \frac{\pi}{2}\right) + 125$$

represents the height $h$, in feet, of a seat on the wheel as a function of time $t$, where $t$ is measured in seconds. The ride begins when $t = 0$.

(a) During the first 40 seconds of the ride, at what time $t$ is an individual on the Ferris wheel exactly 125 feet above the ground?

(b) During the first 80 seconds of the ride, at what time $t$ is an individual on the Ferris wheel exactly 250 feet above the ground?

(c) During the first 40 seconds of the ride, over what interval of time $t$ is an individual on the Ferris wheel more than 125 feet above the ground?

105. **Holding Pattern** An airplane is asked to stay within a holding pattern near Chicago's O'Hare International Airport. The function $d(x) = 70\sin(0.65x) + 150$ represents the distance $d$, in miles, of the airplane from the airport at time $x$, in minutes.

(a) When the plane enters the holding pattern, $x = 0$, how far is it from O'Hare?

(b) During the first 20 minutes after the plane enters the holding pattern, at what time $x$ is the plane exactly 100 miles from the airport?

(c) During the first 20 minutes after the plane enters the holding pattern, at what time $x$ is the plane more than 100 miles from the airport?

(d) While the plane is in the holding pattern, will it ever be within 70 miles of the airport? Why?

**106. Projectile Motion** A golfer hits a golf ball with an initial velocity of 100 miles per hour. The range $R$ of the ball as a function of the angle $\theta$ to the horizontal is given by $R(\theta) = 672\sin(2\theta)$, where $R$ is measured in feet.

(a) At what angle $\theta$ should the ball be hit if the golfer wants the ball to travel 450 feet (150 yards)?

(b) At what angle $\theta$ should the ball be hit if the golfer wants the ball to travel 540 feet (180 yards)?

(c) At what angle $\theta$ should the ball be hit if the golfer wants the ball to travel at least 480 feet (160 yards)?

(d) Can the golfer hit the ball 720 feet (240 yards)?

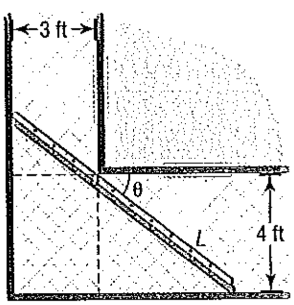
**107. Heat Transfer** In the study of heat transfer, the equation $x + \tan x = 0$ occurs. Graph $Y_1 = -x$ and $Y_2 = \tan x$ for $x \geq 0$. Conclude that there are an infinite number of points of intersection of these two graphs. Now find the first two positive solutions of $x + \tan x = 0$ rounded to two decimal places.

**108. Carrying a Ladder around a Corner** Two hallways, one of width 3 feet, the other of width 4 feet, meet at a right angle. See the illustration. It can be shown that the length $L$ of the ladder as a function of $\theta$ is $L(\theta) = 4\csc\theta + 3\sec\theta$.

(a) In calculus, you are asked to find the length of the longest ladder that can turn the corner by solving the equation

$$3\sec\theta\tan\theta - 4\csc\theta\cot\theta = 0, \quad 0° < \theta < 90°$$

Solve this equation for $\theta$.




(b) What is the length of the longest ladder that can be carried around the corner?

(c) Graph $L = L(\theta), 0° \leq \theta \leq 90°$, and find the angle $\theta$ that minimizes the length $L$.

(d) Compare the result with the one found in part (b). Explain why the two answers are the same.

**109. Projectile Motion** The horizontal distance that a projectile will travel in the air (ignoring air resistance) is given by the equation

$$R(\theta) = \frac{v_0^2\sin(2\theta)}{g}$$

where $v_0$ is the initial velocity of the projectile, $\theta$ is the angle of elevation, and $g$ is acceleration due to gravity (9.8 meters per second squared).

(a) If you can throw a baseball with an initial speed of 34.8 meters per second, at what angle of elevation $\theta$ should you direct the throw so that the ball travels a distance of 107 meters before striking the ground?

(b) Determine the maximum distance that you can throw the ball.

(c) Graph $R = R(\theta)$, with $v_0 = 34.8$ meters per second.

(d) Verify the results obtained in parts (a) and (b) using a graphing utility.

**110. Projectile Motion** Refer to Problem 109.

(a) If you can throw a baseball with an initial speed of 40 meters per second, at what angle of elevation $\theta$ should you direct the throw so that the ball travels a distance of 110 meters before striking the ground?

(b) Determine the maximum distance that you can throw the ball.

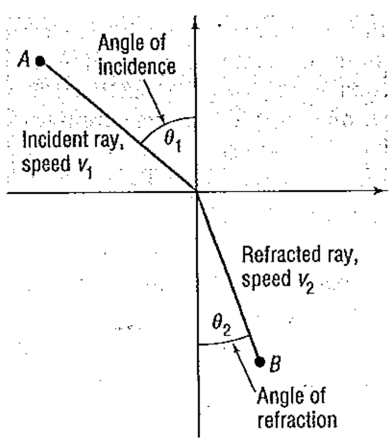
(c) Graph $R = R(\theta)$, with $v_0 = 40$ meters per second.

(d) Verify the results obtained in parts (a) and (b) using a graphing utility.

*The following discussion of Snell's Law of Refraction* (named after Willebrord Snell, 1580–1626) is needed for Problems 111–118. Light, sound, and other waves travel at different speeds, depending on the medium (air, water, wood, and so on) through which they pass. Suppose that light travels from a point A in one medium, where its speed is $v_1$, to a point B in another medium, where its speed is $v_2$. Refer to the figure, where the angle $\theta_1$ is called the angle of incidence and the angle $\theta_2$ is the angle of refraction. Snell's Law, which can be proved using calculus, states that*

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

*The ratio $\frac{v_1}{v_2}$ is called the* index of refraction. *Some values are given in the table.*




**Some Indexes of Refraction**

| Medium | Index of Refraction† |
|---|---|
| Water | 1.33 |
| Ethyl alcohol (20°C) | 1.36 |
| Carbon disulfide | 1.63 |
| Air (1 atm and 0°C) | 1.00029 |
| Diamond | 2.42 |
| Fused quartz | 1.46 |
| Glass, crown | 1.52 |
| Glass, dense flint | 1.66 |
| Sodium chloride | 1.54 |

*Because this law was also deduced by René Descartes in France, it is also known as Descartes' Law.

111. The index of refraction of light in passing from a vacuum into water is 1.33. If the angle of incidence is 40°, determine the angle of refraction.

112. The index of refraction of light in passing from a vacuum into dense flint glass is 1.66. If the angle of incidence is 50°, determine the angle of refraction.

113. Ptolemy, who lived in the city of Alexandria in Egypt during the second century AD, gave the measured values in the following table for the angle of incidence $\theta_1$ and the angle of refraction $\theta_2$ for a light beam passing from air into water. Do these values agree with Snell's Law? If so, what index of refraction results? (These data are of interest as the oldest recorded physical measurements.)

| $\theta_1$ | $\theta_2$ | $\theta_1$ | $\theta_2$ |
|---|---|---|---|
| 10° | 8° | 50° | 35°0′ |
| 20° | 15°30′ | 60° | 40°30′ |
| 30° | 22°30′ | 70° | 45°30′ |
| 40° | 29°0′ | 80° | 50°0′ |

114. **Bending Light** The speed of yellow sodium light (wavelength 589 nanometers) in a certain liquid is measured to be $1.92 \times 10^8$ meters per second. What is the index of refraction of this liquid, with respect to air, for sodium light?*

[**Hint:** The speed of light in air is approximately $2.998 \times 10^8$ meters per second.]

115. **Bending Light** A beam of light with a wavelength of 589 nanometers traveling in air makes an angle of incidence of 40° on a slab of transparent material, and the refracted beam makes an angle of refraction of 26°. Find the index of refraction of the material.*

116. **Bending Light** A light ray with a wavelength of 589 nanometers (produced by a sodium lamp) traveling through air makes an angle of incidence of 30° on a smooth, flat slab of crown glass. Find the angle of refraction.*

117. A light beam passes through a thick slab of material whose index of refraction is $n_2$. Show that the emerging beam is parallel to the incident beam.†

118. **Brewster's Law** If the angle of incidence and the angle of refraction are complementary angles, the angle of incidence is referred to as the Brewster angle $\theta_B$. The Brewster angle is related to the index of refractions of the two media, $n_1$ and $n_2$, by the equation $n_1 \sin \theta_B = n_2 \cos \theta_B$, where $n_1$ is the index of refraction of the incident medium and $n_2$ is the index of refraction of the refractive medium. Determine the Brewster angle for a light beam traveling through water (at 20°C) that makes an angle of incidence with a smooth, flat slab of crown glass.

*For light of wavelength 589 nanometers, measured with respect to a vacuum. The index with respect to air is negligibly different in most cases.

†Adapted from Halliday and Resnick, *Fundamentals of Physics*, 7th ed., 2005, John Wiley & Sons.

## Discussion and Writing

119. Explain in your own words how you would use your calculator to solve the equation $\cos x = -0.6$, $0 \le x < 2\pi$. How would you modify your approach to solve the equation $\cot x = 5$, $0 < x < 2\pi$?

120. Explain why no further points of intersection (and therefore no further solutions) exist in Figure 25 for $x < -\pi$ or $x > 4\pi$.

## Retain Your Knowledge

*Problems 121–124 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.*

121. Convert $6^x = y$ to an equivalent statement involving a logarithm.

122. Find the complex zeros of $f(x) = 2x^2 - 9x + 8$.

123. Given $\sin \theta = -\dfrac{\sqrt{10}}{10}$ and $\cos \theta = \dfrac{3\sqrt{10}}{10}$, find the exact value of each of the four remaining trigonometric functions.

124. Determine the amplitude, period, and phase shift of the function $y = 2 \sin(2x - \pi)$. Graph the function. Show at least two periods.

## 'Are You Prepared?' Answers

1. $\left\{\frac{3}{2}\right\}$ 2. $\frac{\sqrt{2}}{2}; -\frac{1}{2}$ 3. $\left\{-1, \frac{5}{4}\right\}$ 4. $\left\{\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right\}$ 5. $\left\{0, \frac{5}{2}\right\}$ 6. {0.76}

# 6.4 Trigonometric Identities

**PREPARING FOR THIS SECTION** *Before getting started, review the following:*

- Fundamental Identities (Section 5.3, pp. 411–413)
- Even–Odd Properties (Section 5.3, pp. 416–417)

Now Work the 'Are You Prepared?' problems on page 496.

**OBJECTIVES** 1 Use Algebra to Simplify Trigonometric Expressions (p. 493)
2 Establish Identities (p. 493)