Solution Start with the left side and multiply the numerator and the denominator by $1 + \sin\theta$. (Alternatively, we could multiply the numerator and the denominator of the right side by $1 - \sin\theta$.)

$$\frac{1-\sin\theta}{\cos\theta} = \frac{1-\sin\theta}{\cos\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}$$ Multiply the numerator and the denominator by

$$= \frac{1-\sin^2\theta}{\cos\theta(1+\sin\theta)}$$

$$= \frac{\cos^2\theta}{\cos\theta(1+\sin\theta)}$$

$$= \frac{\cos\theta}{1+\sin\theta}$$ Divide out

Now Work PROBLEM 55

Although a lot of practice is the only real way to learn how to establish identities, the following guidelines should prove helpful.

WARNING Be careful not to handle identities to be established as if they were conditional equations. You *cannot* establish an identity by such methods as adding the same expression to each side and obtaining a true statement. This practice is not allowed, because the original statement is precisely the one that you are trying to establish. You do not know until it has been established that it is, in fact, true. ■

Guidelines for Establishing Identities

1. It is almost always preferable to start with the side containing the more complicated expression.
2. Rewrite sums or differences of quotients as a single quotient.
3. Sometimes rewriting one side in terms of sine and cosine functions only will help.
4. Always keep the goal in mind. While manipulating one side of the expression, keep in mind the form of the expression on the other side.

6.4 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. ***True or False*** $\sin^2\theta = 1 - \cos^2\theta$. (p. 412)
2. ***True or False*** $\sin(-\theta) + \cos(-\theta) = \cos\theta - \sin\theta$. (p. 416)

Concepts and Vocabulary

3. Suppose that f and g are two functions with the same domain. If $f(x) = g(x)$ for every x in the domain, the equation is called a(n) _____. Otherwise, it is called a(n) _____ equation.
4. $\tan^2\theta - \sec^2\theta =$ _____.
5. $\cos(-\theta) - \cos\theta =$ _____.
6. ***True or False*** $\sin(-\theta) + \sin\theta = 0$ for any value of θ.
7. ***True or False*** In establishing an identity, it is often easiest to just multiply both sides by a well-chosen nonzero expression involving the variable.
8. ***True or False*** $\tan\theta\cdot\cos\theta = \sin\theta$ for any $\theta \neq (2k+1)\frac{\pi}{2}$.

Skill Building

In Problems 9–18, simplify each trigonometric expression by following the indicated direction.

9. Rewrite in terms of sine and cosine functions:

$$\tan\theta\cdot\csc\theta$$

10. Rewrite in terms of sine and cosine functions:

$$\cot\theta\cdot\sec\theta$$

11. Multiply $\frac{\cos\theta}{1-\sin\theta}$ by $\frac{1+\sin\theta}{1+\sin\theta}$.
12. Multiply $\frac{\sin\theta}{1+\cos\theta}$ by $\frac{1-\cos\theta}{1-\cos\theta}$.
13. Rewrite over a common denominator:

$$\frac{\sin\theta+\cos\theta}{\cos\theta} + \frac{\cos\theta-\sin\theta}{\sin\theta}$$

14. Rewrite over a common denominator:

$$\frac{1}{1-\cos v} + \frac{1}{1+\cos v}$$

15. Multiply and simplify: $\dfrac{(\sin\theta+\cos\theta)(\sin\theta+\cos\theta)-1}{\sin\theta\cos\theta}$

16. Multiply and simplify: $\dfrac{(\tan\theta+1)(\tan\theta+1)-\sec^2\theta}{\tan\theta}$

17. Factor and simplify: $\dfrac{3\sin^2\theta+4\sin\theta+1}{\sin^2\theta+2\sin\theta+1}$

18. Factor and simplify: $\dfrac{\cos^2\theta-1}{\cos^2\theta-\cos\theta}$

In Problems 19–100, establish each identity.

19. $\csc\theta\cdot\cos\theta=\cot\theta$

20. $\sec\theta\cdot\sin\theta=\tan\theta$

21. $1+\tan^2(-\theta)=\sec^2\theta$

22. $1+\cot^2(-\theta)=\csc^2\theta$

23. $\cos\theta(\tan\theta+\cot\theta)=\csc\theta$

24. $\sin\theta(\cot\theta+\tan\theta)=\sec\theta$

25. $\tan u\cot u-\cos^2 u=\sin^2 u$

26. $\sin u\csc u-\cos^2 u=\sin^2 u$

27. $(\sec\theta-1)(\sec\theta+1)=\tan^2\theta$

28. $(\csc\theta-1)(\csc\theta+1)=\cot^2\theta$

29. $(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)=1$

30. $(\csc\theta+\cot\theta)(\csc\theta-\cot\theta)=1$

31. $\cos^2\theta(1+\tan^2\theta)=1$

32. $(1-\cos^2\theta)(1+\cot^2\theta)=1$

33. $(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2=2$

34. $\tan^2\theta\cos^2\theta+\cot^2\theta\sin^2\theta=1$

35. $\sec^4\theta-\sec^2\theta=\tan^4\theta+\tan^2\theta$

36. $\csc^4\theta-\csc^2\theta=\cot^4\theta+\cot^2\theta$

37. $\cos^3 x=\cos x-\sin^2 x\cos x$

38. $\tan^3 x+\tan x=\sec^2 x\tan x$

39. $\sec u-\tan u=\dfrac{\cos u}{1+\sin u}$

40. $\csc u-\cot u=\dfrac{\sin u}{1+\cos u}$

41. $3\sin^2\theta+4\cos^2\theta=3+\cos^2\theta$

42. $9\sec^2\theta-5\tan^2\theta=5+4\sec^2\theta$

43. $1-\dfrac{\cos^2\theta}{1+\sin\theta}=\sin\theta$

44. $1-\dfrac{\sin^2\theta}{1-\cos\theta}=-\cos\theta$

45. $\dfrac{1+\tan v}{1-\tan v}=\dfrac{\cot v+1}{\cot v-1}$

46. $\dfrac{\csc v-1}{\csc v+1}=\dfrac{1-\sin v}{1+\sin v}$

47. $\dfrac{\sec\theta}{\csc\theta}+\dfrac{\sin\theta}{\cos\theta}=2\tan\theta$

48. $\dfrac{\csc\theta-1}{\cot\theta}=\dfrac{\cot\theta}{\csc\theta+1}$

49. $\dfrac{1+\sin\theta}{1-\sin\theta}=\dfrac{\csc\theta+1}{\csc\theta-1}$

50. $\dfrac{\cos\theta+1}{\cos\theta-1}=\dfrac{1+\sec\theta}{1-\sec\theta}$

51. $\dfrac{1-\sin v}{\cos v}+\dfrac{\cos v}{1-\sin v}=2\sec v$

52. $\dfrac{\cos v}{1+\sin v}+\dfrac{1+\sin v}{\cos v}=2\sec v$

53. $\dfrac{\sin\theta}{\sin\theta-\cos\theta}=\dfrac{1}{1-\cot\theta}$

54. $1-\dfrac{\sin^2\theta}{1+\cos\theta}=\cos\theta$

55. $\dfrac{1-\sin\theta}{1+\sin\theta}=(\sec\theta-\tan\theta)^2$

56. $\dfrac{1-\cos\theta}{1+\cos\theta}=(\csc\theta-\cot\theta)^2$

57. $\dfrac{\cos\theta}{1-\tan\theta}+\dfrac{\sin\theta}{1-\cot\theta}=\sin\theta+\cos\theta$

58. $\dfrac{\cot\theta}{1-\tan\theta}+\dfrac{\tan\theta}{1-\cot\theta}=1+\tan\theta+\cot\theta$

59. $\tan\theta+\dfrac{\cos\theta}{1+\sin\theta}=\sec\theta$

60. $\dfrac{\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}=\dfrac{\tan\theta}{1-\tan^2\theta}$

61. $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=\tan\theta+\sec\theta$

62. $\dfrac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1}=\dfrac{\sin\theta+1}{\cos\theta}$

63. $\dfrac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta}=\sin^2\theta-\cos^2\theta$

64. $\dfrac{\sec\theta-\cos\theta}{\sec\theta+\cos\theta}=\dfrac{\sin^2\theta}{1+\cos^2\theta}$

65. $\dfrac{\tan u-\cot u}{\tan u+\cot u}+1=2\sin^2 u$

66. $\dfrac{\tan u-\cot u}{\tan u+\cot u}+2\cos^2 u=1$

67. $\dfrac{\sec\theta+\tan\theta}{\cot\theta+\cos\theta}=\tan\theta\sec\theta$

68. $\dfrac{\sec\theta}{1+\sec\theta}=\dfrac{1-\cos\theta}{\sin^2\theta}$

69. $\dfrac{1-\tan^2\theta}{1+\tan^2\theta}+1=2\cos^2\theta$

70. $\dfrac{1-\cot^2\theta}{1+\cot^2\theta}+2\cos^2\theta=1$

71. $\dfrac{\sec\theta-\csc\theta}{\sec\theta\csc\theta}=\sin\theta-\cos\theta$

72. $\dfrac{\sin^2\theta-\tan\theta}{\cos^2\theta-\cot\theta}=\tan^2\theta$

73. $\sec\theta-\cos\theta=\sin\theta\tan\theta$

74. $\tan\theta+\cot\theta=\sec\theta\csc\theta$

75. $\dfrac{1}{1-\sin\theta}+\dfrac{1}{1+\sin\theta}=2\sec^2\theta$

76. $\dfrac{1+\sin\theta}{1-\sin\theta}-\dfrac{1-\sin\theta}{1+\sin\theta}=4\tan\theta\sec\theta$

77. $\dfrac{\sec\theta}{1-\sin\theta}=\dfrac{1+\sin\theta}{\cos^3\theta}$

78. $\dfrac{1+\sin\theta}{1-\sin\theta}=(\sec\theta+\tan\theta)^2$

79. $\dfrac{(\sec v-\tan v)^2+1}{\csc v(\sec v-\tan v)}=2\tan v$

80. $\dfrac{\sec^2 v-\tan^2 v+\tan v}{\sec v}=\sin v+\cos v$

81. $\dfrac{\sin\theta+\cos\theta}{\cos\theta}-\dfrac{\sin\theta-\cos\theta}{\sin\theta}=\sec\theta\csc\theta$

82. $\dfrac{\sin\theta+\cos\theta}{\sin\theta}-\dfrac{\cos\theta-\sin\theta}{\cos\theta}=\sec\theta\csc\theta$

83. $\dfrac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}=1-\sin\theta\cos\theta$

84. $\dfrac{\sin^3\theta+\cos^3\theta}{1-2\cos^2\theta}=\dfrac{\sec\theta-\sin\theta}{\tan\theta-1}$

85. $\dfrac{\cos^2\theta-\sin^2\theta}{1-\tan^2\theta}=\cos^2\theta$

86. $\dfrac{\cos\theta+\sin\theta-\sin^3\theta}{\sin\theta}=\cot\theta+\cos^2\theta$

87. $\dfrac{(2\cos^2\theta-1)^2}{\cos^4\theta-\sin^4\theta}=1-2\sin^2\theta$

88. $\dfrac{1-2\cos^2\theta}{\sin\theta\cos\theta}=\tan\theta-\cot\theta$

89. $\dfrac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta}=\dfrac{1+\cos\theta}{\sin\theta}$

90. $\dfrac{1+\cos\theta+\sin\theta}{1+\cos\theta-\sin\theta}=\sec\theta+\tan\theta$

91. $(a \sin\theta + b\cos\theta)^2 + (a\cos\theta - b\sin\theta)^2 = a^2 + b^2$

92. $(2a\sin\theta\cos\theta)^2 + a^2(\cos^2\theta - \sin^2\theta)^2 = a^2$

93. $\dfrac{\tan\alpha + \tan\beta}{\cot\alpha + \cot\beta} = \tan\alpha\tan\beta$

94. $(\tan\alpha + \tan\beta)(1 - \cot\alpha\cot\beta) + (\cot\alpha + \cot\beta)(1 - \tan\alpha\tan\beta) = 0$

95. $(\sin\alpha + \cos\beta)^2 + (\cos\beta + \sin\alpha)(\cos\beta - \sin\alpha) = 2\cos\beta(\sin\alpha + \cos\beta)$

96. $(\sin\alpha - \cos\beta)^2 + (\cos\beta + \sin\alpha)(\cos\beta - \sin\alpha) = -2\cos\beta(\sin\alpha - \cos\beta)$

97. $\ln|\sec\theta| = -\ln|\cos\theta|$

98. $\ln|\tan\theta| = \ln|\sin\theta| - \ln|\cos\theta|$

99. $\ln|1 + \cos\theta| + \ln|1 - \cos\theta| = 2\ln|\sin\theta|$

100. $\ln|\sec\theta + \tan\theta| + \ln|\sec\theta - \tan\theta| = 0$

In Problems 101–104, show that the functions f and g are identically equal.

101. $f(x) = \sin x \cdot \tan x \quad g(x) = \sec x - \cos x$

102. $f(x) = \cos x \cdot \cot x \quad g(x) = \csc x - \sin x$

103. $f(\theta) = \dfrac{1 - \sin\theta}{\cos\theta} - \dfrac{\cos\theta}{1 + \sin\theta} \quad g(\theta) = 0$

104. $f(\theta) = \tan\theta + \sec\theta \quad g(\theta) = \dfrac{\cos\theta}{1 - \sin\theta}$

105. Show $\sqrt{16 + 16\tan^2\theta} = 4\sec\theta$ if $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

106. Show $\sqrt{9\sec^2\theta - 9} = 3\tan\theta$ if $\pi \le \theta < \dfrac{3\pi}{2}$.

Applications and Extensions

107. Searchlights A searchlight at the grand opening of a new car dealership casts a spot of light on a wall located 75 meters from the searchlight. The acceleration $\ddot{r}$ of the spot of light is found to be $\ddot{r} = 1200\sec\theta(2\sec^2\theta - 1)$. Show that this is equivalent to $\ddot{r} = 1200\left(\dfrac{1 + \sin^2\theta}{\cos^3\theta}\right)$.

Source: Adapted from Hibbeler, *Engineering Mechanics: Dynamics,* 10th ed., Prentice Hall © 2004.

108. Optical Measurement Optical methods of measurement often rely on the interference of two light waves. If two light waves, identical except for a phase lag, are mixed together, the resulting intensity, or irradiance, is given by $I_t = 4A^2\dfrac{(\csc\theta - 1)(\sec\theta + \tan\theta)}{\csc\theta\sec\theta}$. Show that this is equivalent to $I_t = (2A\cos\theta)^2$.

Source: Experimental Techniques, July/August 2002

Discussion and Writing

109. Write a few paragraphs outlining your strategy for establishing identities.

110. Write down the three Pythagorean Identities.

111. Why do you think it is usually preferable to start with the side containing the more complicated expression when establishing an identity?

112. Make up an identity that is not a Fundamental Identity.

Retain Your Knowledge

Problems 113–116 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

113. Determine whether $f(x) = -3x^2 + 120x + 50$ has a maximum or a minimum value, and then find the value.

114. Given $f(x) = \dfrac{x + 1}{x - 2}$ and $g(x) = 3x - 4$, find $f \circ g$.

115. Find the exact values of the six trigonometric functions of an angle θ if $(-12, 5)$ is a point on its terminal side in standard position.

116. Find the average rate of change of $f(x) = \cos x$ from 0 to $\dfrac{\pi}{2}$.

'Are You Prepared?' Answers

1. True **2.** True