With this formula, the solution to Example 7(c) can be obtained as follows:

$$\cos\alpha = -\frac{3}{5} \quad \pi < \alpha < \frac{3\pi}{2}$$

$$\sin\alpha = -\sqrt{1-\cos^2\alpha} = -\sqrt{1-\frac{9}{25}} = -\sqrt{\frac{16}{25}} = -\frac{4}{5}$$

Then, by equation (11),

$$\tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\left(-\frac{3}{5}\right)}{-\frac{4}{5}} = \frac{\frac{8}{5}}{-\frac{4}{5}} = -2$$

6.6 Assess Your Understanding

Concepts and Vocabulary

1. $\cos(2\theta) = \cos^2\theta -$ ____ $=$ ____ $- 1 = 1 -$ ____.

2. $\sin^2\frac{\theta}{2} = \frac{\_\_\_\_}{2}$

3. $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\_\_\_\_}$

4. ***True or False*** $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

5. ***True or False*** $\sin(2\theta)$ has two equivalent forms:

   $2\sin\theta\cos\theta$ and $\sin^2\theta - \cos^2\theta$

6. ***True or False*** $\tan(2\theta) + \tan(2\theta) = \tan(4\theta)$

Skill Building

In Problems 7–18, use the information given about the angle θ, $0 \le \theta < 2\pi$, to find the exact value of:

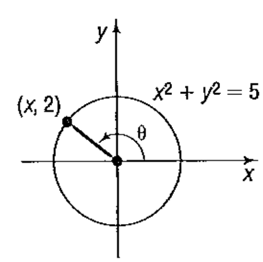
(a) $\sin(2\theta)$ (b) $\cos(2\theta)$ (c) $\sin\frac{\theta}{2}$ (d) $\cos\frac{\theta}{2}$ (e) $\tan(2\theta)$ (f) $\tan\frac{\theta}{2}$

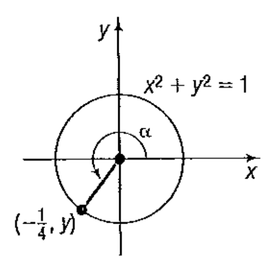
7. $\sin\theta = \frac{3}{5}$, $0 < \theta < \frac{\pi}{2}$
8. $\cos\theta = \frac{3}{5}$, $0 < \theta < \frac{\pi}{2}$
9. $\tan\theta = \frac{4}{3}$, $\pi < \theta < \frac{3\pi}{2}$
10. $\tan\theta = \frac{1}{2}$, $\pi < \theta < \frac{3\pi}{2}$
11. $\cos\theta = -\frac{\sqrt{6}}{3}$, $\frac{\pi}{2} < \theta < \pi$
12. $\sin\theta = -\frac{\sqrt{3}}{3}$, $\frac{3\pi}{2} < \theta < 2\pi$
13. $\sec\theta = 3$, $\sin\theta > 0$
14. $\csc\theta = -\sqrt{5}$, $\cos\theta < 0$
15. $\cot\theta = -2$, $\sec\theta < 0$
16. $\sec\theta = 2$, $\csc\theta < 0$
17. $\tan\theta = -3$, $\sin\theta < 0$
18. $\cot\theta = 3$, $\cos\theta < 0$

In Problems 19–28, use the Half-angle Formulas to find the exact value of each expression.

19. $\sin 22.5°$
20. $\cos 22.5°$
21. $\tan\frac{7\pi}{8}$
22. $\tan\frac{9\pi}{8}$
23. $\cos 165°$
24. $\sin 195°$
25. $\sec\frac{15\pi}{8}$
26. $\csc\frac{7\pi}{8}$
27. $\sin\left(-\frac{\pi}{8}\right)$
28. $\cos\left(-\frac{3\pi}{8}\right)$

In Problems 29–40, use the figures to evaluate each function given that $f(x) = \sin x$, $g(x) = \cos x$, and $h(x) = \tan x$.





29. $f(2\theta)$
30. $g(2\theta)$
31. $g\left(\frac{\theta}{2}\right)$
32. $f\left(\frac{\theta}{2}\right)$
33. $h(2\theta)$
34. $h\left(\frac{\theta}{2}\right)$
35. $g(2\alpha)$
36. $f(2\alpha)$
37. $f\left(\frac{\alpha}{2}\right)$
38. $g\left(\frac{\alpha}{2}\right)$
39. $h\left(\frac{\alpha}{2}\right)$
40. $h(2\alpha)$

41. Show that $\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)$.
42. Show that $\sin(4\theta) = (\cos\theta)(4\sin\theta - 8\sin^3\theta)$.
43. Develop a formula for $\cos(3\theta)$ as a third-degree polynomial in the variable $\cos\theta$.
44. Develop a formula for $\cos(4\theta)$ as a fourth-degree polynomial in the variable $\cos\theta$.
45. Find an expression for $\sin(5\theta)$ as a fifth-degree polynomial in the variable $\sin\theta$.
46. Find an expression for $\cos(5\theta)$ as a fifth-degree polynomial in the variable $\cos\theta$.

In Problems 47–68, establish each identity.

47. $\cos^4\theta - \sin^4\theta = \cos(2\theta)$
48. $\frac{\cot\theta - \tan\theta}{\cot\theta + \tan\theta} = \cos(2\theta)$
49. $\cot(2\theta) = \frac{\cot^2\theta - 1}{2\cot\theta}$
50. $\cot(2\theta) = \frac{1}{2}(\cot\theta - \tan\theta)$
51. $\sec(2\theta) = \frac{\sec^2\theta}{2 - \sec^2\theta}$
52. $\csc(2\theta) = \frac{1}{2}\sec\theta\csc\theta$
53. $\cos^2(2u) - \sin^2(2u) = \cos(4u)$
54. $(4\sin u\cos u)(1 - 2\sin^2 u) = \sin(4u)$
55. $\frac{\cos(2\theta)}{1 + \sin(2\theta)} = \frac{\cot\theta - 1}{\cot\theta + 1}$
56. $\sin^2\theta\cos^2\theta = \frac{1}{8}[1 - \cos(4\theta)]$
57. $\sec^2\frac{\theta}{2} = \frac{2}{1 + \cos\theta}$
58. $\csc^2\frac{\theta}{2} = \frac{2}{1 - \cos\theta}$
59. $\cot^2\frac{v}{2} = \frac{\sec v + 1}{\sec v - 1}$
60. $\tan\frac{v}{2} = \csc v - \cot v$
61. $\cos\theta = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}}$
62. $1 - \frac{1}{2}\sin(2\theta) = \frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta}$
63. $\frac{\sin(3\theta)}{\sin\theta} - \frac{\cos(3\theta)}{\cos\theta} = 2$
64. $\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} - \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = 2\tan(2\theta)$
65. $\tan(3\theta) = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$
66. $\tan\theta + \tan(\theta + 120°) + \tan(\theta + 240°) = 3\tan(3\theta)$
67. $\ln|\sin\theta| = \frac{1}{2}(\ln|1 - \cos(2\theta)| - \ln 2)$
68. $\ln|\cos\theta| = \frac{1}{2}(\ln|1 + \cos(2\theta)| - \ln 2)$

In Problems 69–78, solve each equation on the interval $0 \le \theta < 2\pi$.

69. $\cos(2\theta) + 6\sin^2\theta = 4$
70. $\cos(2\theta) = 2 - 2\sin^2\theta$
71. $\cos(2\theta) = \cos\theta$
72. $\sin(2\theta) = \cos\theta$
73. $\sin(2\theta) + \sin(4\theta) = 0$
74. $\cos(2\theta) + \cos(4\theta) = 0$
75. $3 - \sin\theta = \cos(2\theta)$
76. $\cos(2\theta) + 5\cos\theta + 3 = 0$
77. $\tan(2\theta) + 2\sin\theta = 0$
78. $\tan(2\theta) + 2\cos\theta = 0$

Mixed Practice

In Problems 79–90, find the exact value of each expression.

79. $\sin\left(2\sin^{-1}\frac{1}{2}\right)$
80. $\sin\left[2\sin^{-1}\frac{\sqrt{3}}{2}\right]$
81. $\cos\left(2\sin^{-1}\frac{3}{5}\right)$
82. $\cos\left(2\cos^{-1}\frac{4}{5}\right)$
83. $\tan\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right]$
84. $\tan\left(2\tan^{-1}\frac{3}{4}\right)$
85. $\sin\left(2\cos^{-1}\frac{4}{5}\right)$
86. $\cos\left[2\tan^{-1}\left(-\frac{4}{3}\right)\right]$
87. $\sin^2\left(\frac{1}{2}\cos^{-1}\frac{3}{5}\right)$
88. $\cos^2\left(\frac{1}{2}\sin^{-1}\frac{3}{5}\right)$
89. $\sec\left(2\tan^{-1}\frac{3}{4}\right)$
90. $\csc\left[2\sin^{-1}\left(-\frac{3}{5}\right)\right]$

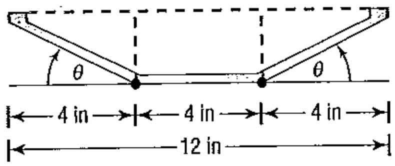
In Problems 91–93, find the real zeros of each trigonometric function on the interval $0 \le \theta < 2\pi$.

91. $f(x) = \sin(2x) - \sin x$
92. $f(x) = \cos(2x) + \cos x$
93. $f(x) = \cos(2x) + \sin^2 x$

Applications and Extensions

94. Constructing a Rain Gutter A rain gutter is to be constructed of aluminum sheets 12 inches wide. After marking off a length of 4 inches from each edge, the builder bends this length up at an angle θ. See the illustration. The area A of the opening as a function of θ is given by

$$A(\theta) = 16 \sin\theta(\cos\theta + 1) \quad 0° < \theta < 90°$$



(a) In calculus, you will be asked to find the angle θ that maximizes A by solving the equation

$$\cos(2\theta) + \cos\theta = 0, \quad 0° < \theta < 90°$$

Solve this equation for θ.

(b) What is the maximum area A of the opening?

(c) Graph $A = A(\theta), 0° \le \theta \le 90°$, and find the angle θ that maximizes the area A. Also find the maximum area. Compare the results to the answer found earlier.

95. Laser Projection In a laser projection system, the **optical angle** or **scanning angle** θ is related to the throw distance D from the scanner to the screen and the projected image width W by the equation

$$D = \frac{\frac{1}{2}W}{\csc\theta - \cot\theta}$$

(a) Show that the projected image width is given by

$$W = 2D\tan\frac{\theta}{2}$$

(b) Find the optical angle if the throw distance is 15 feet and the projected image width is 6.5 feet.

Source: Pangolin Laser Systems, Inc.

96. Product of Inertia The **product of inertia** for an area about inclined axes is given by the formula

$$I_{uv} = I_x \sin\theta\cos\theta - I_y \sin\theta\cos\theta + I_{xy}(\cos^2\theta - \sin^2\theta)$$

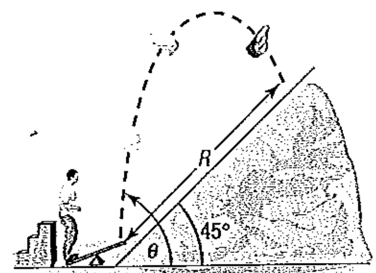
Show that this is equivalent to

$$I_{uv} = \frac{I_x - I_y}{2}\sin(2\theta) + I_{xy}\cos(2\theta)$$

Source: Adapted from Hibbeler, *Engineering Mechanics: Statics*, 10th ed., Prentice Hall © 2004.

97. Projectile Motion An object is propelled upward at an angle θ, $45° < \theta < 90°$, to the horizontal with an initial velocity of v_0 feet per second from the base of a plane that makes an angle of 45° with the horizontal. See the illustration. If air resistance is ignored, the distance R that it travels up the inclined plane is given by the function

$$R(\theta) = \frac{v_0^2\sqrt{2}}{16}\cos\theta(\sin\theta - \cos\theta)$$



(a) Show that

$$R(\theta) = \frac{v_0^2\sqrt{2}}{32}[\sin(2\theta) - \cos(2\theta) - 1]$$

(b) In calculus, you will be asked to find the angle θ that maximizes R by solving the equation

$$\sin(2\theta) + \cos(2\theta) = 0$$

Solve this equation for θ.

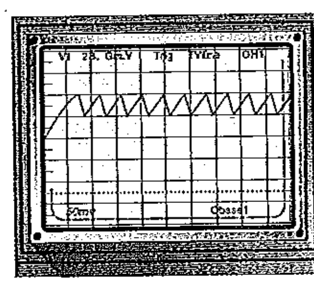
(c) What is the maximum distance R if $v_0 = 32$ feet per second?

(d) Graph $R = R(\theta), 45° \le \theta \le 90°$, and find the angle θ that maximizes the distance R. Also find the maximum distance. Use $v_0 = 32$ feet per second. Compare the results with the answers found earlier.

98. Sawtooth Curve An oscilloscope often displays a sawtooth curve. This curve can be approximated by sinusoidal curves of varying periods and amplitudes. A first approximation to the sawtooth curve is given by

$$y = \frac{1}{2}\sin(2\pi x) + \frac{1}{4}\sin(4\pi x)$$

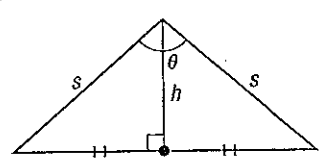
Show that $y = \sin(2\pi x)\cos^2(\pi x)$.



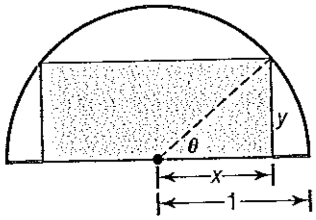
99. Area of an Isosceles Triangle Show that the area A of an isosceles triangle whose equal sides are of length s, and where θ is the angle between them, is

$$A = \frac{1}{2}s^2\sin\theta$$

[**Hint:** See the illustration. The height h bisects the angle θ and is the perpendicular bisector of the base.]



100. **Geometry** A rectangle is inscribed in a semicircle of radius 1. See the illustration.



(a) Express the area A of the rectangle as a function of the angle θ shown in the illustration.

(b) Show that $A(\theta) = \sin(2\theta)$.

(c) Find the angle θ that results in the largest area A.

(d) Find the dimensions of this largest rectangle.

101. If $x = 2\tan\theta$, express $\sin(2\theta)$ as a function of x.

102. If $x = 2\tan\theta$, express $\cos(2\theta)$ as a function of x.

103. Find the value of the number C:

$$\frac{1}{2}\sin^2 x + C = -\frac{1}{4}\cos(2x)$$

104. Find the value of the number C:

$$\frac{1}{2}\cos^2 x + C = \frac{1}{4}\cos(2x)$$

105. If $z = \tan\frac{\alpha}{2}$, show that $\sin\alpha = \frac{2z}{1+z^2}$.

106. If $z = \tan\frac{\alpha}{2}$, show that $\cos\alpha = \frac{1-z^2}{1+z^2}$.

107. Graph $f(x) = \sin^2 x = \frac{1-\cos(2x)}{2}$ for $0 \le x \le 2\pi$ by hand using transformations.

108. Repeat Problem 107 for $g(x) = \cos^2 x$.

109. Use the fact that

$$\cos\frac{\pi}{12} = \frac{1}{4}(\sqrt{6} + \sqrt{2})$$

to find $\sin\frac{\pi}{24}$ and $\cos\frac{\pi}{24}$.

110. Show that

$$\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

and use it to find $\sin\frac{\pi}{16}$ and $\cos\frac{\pi}{16}$.

111. Show that

$$\sin^3\theta + \sin^3(\theta + 120°) + \sin^3(\theta + 240°) = -\frac{3}{4}\sin(3\theta)$$

112. If $\tan\theta = a\tan\frac{\theta}{3}$, express $\tan\frac{\theta}{3}$ in terms of a.

113. For $\cos(2x) + (2m - 1)\sin x + m - 1 = 0$, find m such that there is exactly one real solution for x, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.†

Discussion and Writing

114. Go to the library and research Chebyshëv polynomials. Write a report on your findings.

Retain Your Knowledge

Problems 115–118 are based on material learned earlier in the course. The purpose of these problems is to keep the material fresh in your mind so that you are better prepared for the final exam.

115. Find the equation of the line that contains the point $(2, -3)$ and is perpendicular to the graph of $y = -2x + 9$.

116. Graph $f(x) = -x^2 + 6x + 7$. Label the vertex and any intercepts.

117. Find the exact value of $\sin\left(\frac{2\pi}{3}\right) - \cos\left(\frac{4\pi}{3}\right)$.

118. Graph $y = -2\cos\left(\frac{\pi}{2}x\right)$. Show at least two periods.

6.7 Product-to-Sum and Sum-to-Product Formulas

OBJECTIVES
1 Express Products as Sums (p. 521)
2 Express Sums as Products (p. 523)

1 Express Products as Sums

Sum and difference formulas can be used to derive formulas for writing the products of sines and/or cosines as sums or differences. These identities are usually called the **Product-to-Sum Formulas.**

†Courtesy of Joliet Junior College Mathematics Department